Generation of Fronts by Mixing and Mutual Intrusion

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ABSTRACT

In this paper a mechanism is proposed which could be responsible for the formation of sharp horizontal density gradients such as those observed in shallow seas, from fluid which initially has weak horizontal density gradients. The sharp density gradients result from the mutual intrusion of several stratified bodies of water which were exposed to various degrees of vertical mixing for a limited amount of time. The dynamics of the intrusion are examined by a simplified nonrotating, frictionless multilayer model. The results are compared quantitatively to laboratory experiments and qualitatively to field observations.

The theoretical model contains an upper and lower portion, each of which consists of several bodies of fluids with different densities corresponding to various degrees of mixing. It predicts that in both the upper and the lower portions, fluids which were exposed to intermediate mixing sink rapidly from the surface, rise from the bottom, and after a finite amount of time concentrate in mid-depth. This results in a formation of density discontinuities (fronts) near the surface, bottom, and in the boundary between the upper and the lower portions.

Rotation is excluded from the simplified model, but it is expected that mutual intrusion will take place even if rotation is included, provided that the flow is not in an exact geostrophic balance. The theoretical predictions were tested in the laboratory in a tank which contained several bodies of water with different densities separated initially by a number of gates. The experimental results compare favorably with the theoretical predictions. Observations which suggest the existence of mutual intrusion in frontal zones are discussed.

1. Introduction

Fronts represent regions of extremely high horizontal gradients of temperature or salinity (or both) and their importance as a physical process in the ocean has been summarized by Mooers (1978). Recent studies have drawn attention to frontal structures occurring in estuaries (Klemas and Polis, 1977) and on the continental shelves (Simpson et al., 1978; James, 1977).

Estuarine and shelf fronts differ from each other in several respects. Estuarine fronts are usually associated with sharp salinity gradients and may have a length scale of 1-10 km, while shelf fronts are usually associated with strong temperature gradients and may have a length scale of 100 km. Although there are differences between these two classes of fronts, both are reported to be associated with regions of shoaling and tidal motion (see, e.g., Simpson and Hunter, 1974; Officer, 1976). Simpson and Hunter (1974) pointed out that the location of shelf fronts is associated with a depth where the parameter $h/\nu^3$ (where $h$ is the water depth and $\nu$ the amplitude of the tidal current) reaches a certain value. They suggested that the fronts are produced by tidal mixing and that at the frontal boundary there is an abrupt transition between a vertically mixed domain on the shallow region and stratified on the deep. For such fronts Fearnhead (1975) derived a "stratification parameter" from consideration of the tidal power required to mix a given body of fluid.

An example of a shelf front is given in Fig. 1, which shows that the front consists of a sharp horizontal density gradient and that a bottom front is associated with the surface front. A similar density structure has been recently observed in the Haro Strait (British Columbia), whose main circulation is of the estuarine type. Since both shelf and estuarine fronts are associated with areas of shoaling and both may have a density structure similar to the one shown in Fig. 1, it is possible that they are generated by similar mechanisms.

Typical frontal widths are $\sim 1$ km for shelf fronts and a few meters for estuarine fronts, while the bottom slope length scale ($L$, Fig. 2) is usually much larger, typically $\sim 10$ km and $\sim 1$ km for shelf and estuarine fronts, respectively. That is, for both classes of fronts, the bottom slope length scale is

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2 These observations are presently being analyzed and will be reported elsewhere.
at least an order of magnitude larger than the frontal width. This raises the fundamental question regarding the possibility of the mixing mechanism changing abruptly over a gradual shoaling bottom. To date, it has not been demonstrated that the mixing mechanism may suffer an abrupt change over the front width and it is questionable whether such changes occur in nature. Schumacher et al. (1978) suggest that the obvious forcing functions are unlikely to yield strong gradients over the front width and therefore do not explain frontogenesis. In this paper we address the question whether it is necessary for the mixing mechanism to change abruptly in order to generate a front. We shall show that even if the mixing mechanism varies gradually over the bottom slope, a front will be generated by mutual intrusion. To show this, we shall assume a gradually varying mixing mechanism and examine the flow which results from the energy input to the system. We shall focus our attention on the processes which may generate sharp horizontal density gradients; no attempt will be made to examine the processes which maintain the fronts during their lifetime nor to examine their decay.

Before describing our simplified model, it should be mentioned that the generation mechanism of the two categories of fronts considered here (estuarine and shelf) differs probably from the generation mechanism of fronts which result from a river emptying into an open coastal area. Such river plume fronts were studied by Kao et al. (1977) and Garvine (1974). These fronts are generated by a supply of freshwater from a source which is independent of the ocean, while estuarine and shelf fronts are produced in the absence of such an independent source. For some estuarine fronts, it is difficult to determine whether they are generated by a freshwater discharge or another mechanism. For instance, the observations of Ingram (1976) in the St. Lawrence estuary were located near a freshwater source and it is not clear from his results to which class they belong.

We shall now describe our simplified model. We assume an initial density structure formed by gradually varying mixing (see Figs. 2 and 3a) and examine the flow which results from the increase of potential energy in the region where strong mixing acted on the system. It is difficult to determine the appropriate period of time required to form the initial density structure. However, if we take the vertical eddy viscosity coefficient (associated with a high-amplitude strong tidal current) to be $\nu \approx 500 \text{ cm}^2 \text{s}^{-1}$ and the depth of the upper layer to be $d = 20 \text{ m}$, we find that such a structure can be formed within $t = O(d^2/\nu) \approx 2 \text{ h}$.

It is also assumed that the gradually varying mixing mechanism relaxes after the density structure (shown in Fig. 3a) has been formed. This relaxation corresponds to the low velocities which occur during the tidal cycle and is supported by the idea that the mixing mechanism is proportional to $\nu^3$ and hence is relatively weak during some portions of the cycle. The effect of the sloping bottom is taken into account only as far as it affects the mixing. The neglect of the dynamical effect of a sloping bottom is con-

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**Fig. 1.** Section through a front in the Celtic Sea, 23–24 June 1973 (adapted from James, 1977). The region of shoaling which is responsible for the front formation is located slightly to the northeast and northwest of this section and therefore is not apparent in the bottom profile.

**Fig. 2.** Schematic diagram of the assumed density structure prior to any vertical mixing.
sidered justified in view of Middleton's (1966) experiments on gravity currents, which show that small slopes play only secondary roles. However, it is expected that our approximation will break down for steep slopes. We do not consider the advection due to local currents but rather suppose that the whole field in question is not fixed in space but moves bodily with the local currents.

Initially, after the mixing has been completed, the fluid is strongly stratified on one side and the degree of stratification decreases gradually toward the region which experienced relatively strong mixing. In the absence of rotation, the pressure gradient force, which was created by the mixing, will drive a flow in which the mixed water intrudes into the stratified and vice versa.

The problem is simplified by representing waters which were exposed to intermediate mixing by two bodies of water (Fig. 3b). Assuming that the Reynolds and the Péclet numbers (based on eddy viscosity and diffusivity values, rather than molecular values) are of order several hundred, we consider a frictionless nondiffusive model. The problem is further reduced to a three-layer model by dividing the domain into an upper and lower portion, solving the equations in each, and matching the solutions at the dividing line. More elaborate and realistic models could be considered, but it is believed that our model will serve adequately to illustrate the fundamental points under discussion.

The Boussinesq and the rigid-lid approximation are invoked, but in regions where large vertical velocities are expected the flow is not constrained to be hydrostatic. For the three-layer model, the integrated form of the momentum and the energy equation are applied in a moving coordinate system. For this frame of reference, the governing equations yield a pair of high-order polynomials which are solved numerically. Two solutions are found in the physical range and the non-uniqueness question is resolved by considering the history and future of the flow. These theoretical results are presented and discussed in Sections 3 and 4.

Laboratory experiments were performed in a long and narrow tank which contained several bodies of saltwater separated initially by a number of gates. The mutual intrusion was photographed to obtain measurements of the velocities and the depth of the various layers. These experimental results are discussed in Section 5. Section 6 summarizes this work and discusses its applications.

2. Formulation

We consider a stratified region, as shown in Fig. 3a. The domain on the right-hand side represents the region where no vertical mixing took place, the region on the left represents the domain where vigorous mixing acted on the system. As mentioned earlier, it is assumed that the degree of mixing varied gradually from very weak on the right to very strong on the left. Under such conditions, the density on the far left ($\rho_2$) equals $(\rho_1 H + \rho_3 \bar{H}) (H + \bar{H})^{-1}$, where $\rho_1$ and $\rho_3$ are the densities of the upper and lower layer on the right side and $H$ and $\bar{H}$ are the depths of the upper and lower layers, respectively.
Since we are mainly interested in the formation of horizontal density gradients, it is expected that our two-layer approximation of the vertical density field will not affect the solution substantially, although the details of the intrusion will be altered. We shall return to the effect of this approximation in Section 4.

To simplify our analysis, we divide the region into six domains as shown in Fig. 3b. Both the upper and lower portions (the regions above and below the line connecting points 3 and 4) include three bodies of water with different densities. In the upper portion, the density increases from right to left, while in the lower the density increases from left to right. With this simplification, the regions where intermediate mixing took place are represented by two bodies of water with intermediate densities. It will be shown later that laboratory experiments suggest that no loss of generality is introduced to the problem by this simplification.

For convenience, we choose the density of the intermediate water in the upper portion to be \(\rho\), while the fluids to its left (\(\rho_1\)) and right (\(\rho_2\)) have a density of \((\rho + \Delta\rho)\) and \((\rho - \Delta\rho)\), respectively. The densities of the lower portion are such that, initially, after the mixing has been completed but before any motion, the total mass of a fluid column does not vary in the field as required by continuity. Such considerations lead to densities of \((\rho + \Delta\rho)\), \(\rho + \Delta\rho(1 + H/H)\) and \(\rho + \Delta\rho(1 + 2H/H)\) on the left, central and right domains, respectively.

Since the dividing line is expected to be some kind of a symmetry line, we may replace it with a solid bottom, solve the problem in the upper and lower portion separately, and match the solutions at the dividing line. Due to the exclusion of friction, the matching condition is continuity of pressure alone. It will become clear later, however, that the Bernoulli principle yields that a continuity of the velocities ensures the continuity of pressure.

3. Three-layer model

We shall consider now the upper portion of the simplified density structure. The problem is equivalent to the mutual intrusion of three bodies of water in a tank separated initially by two gates. After the gates are lifted, one expects to find the following major stages of intrusion.

a. First stage

In this early stage, the intrusion near each gate does not feel the presence of the other due to the distance between the gates (see Fig. 4). Thus, the solution is given simply by the solution to the mutual intrusion of two fluids. This was given by Yih (1947) and Benjamin (1968) who both found that for a non-dissipating irrotational flow the velocity of the intrusion \(c_1\) is

\[
c_1 = \frac{1}{2} g (\Delta \rho / \rho) H^{1/2},
\]

where \(g\) is the acceleration of gravity and \(H\) the total depth of the fluid column. The structure of the intrusion is such that each fluid occupies half the depth. For \((\Delta \rho / \rho) = 10^{-3}\) and a depth of 50 m, (1) gives \(\sim 0.35\) m s\(^{-1}\) as a typical intruding velocity in the ocean.

After some time the interface near the center of the tank would collide if (1) completely describes the flow. Therefore, (1) would cease to be valid for the central region and an intermediate stage with a three-layer system in the central region will be established (see Fig. 5, which shows the left and central regions of the tank).

b. Intermediate stage

In order to find the solution for this case, we assume that the flow in the far field, near point 1, remains essentially unaltered; that is to say, that the velocity for this region is given by (1). This assumption is equivalent to Long's hypothesis of no upstream influence (see, e.g., Long, 1972), since if we view the flow from a coordinate system which travels to the left with point 2, the field resembles a flow of two layers below an obstacle. The validity of Long's hypothesis has been a controversial issue for many years (Baines, 1977; Turner, 1973), but we shall show that laboratory experiments suggest that for the case under consideration the assumption is probably adequate. Even if there is an upstream influence, it is weak and it is possible to investigate the problem disregarding the changes in the far field.

![Fig. 4. First intrusion stage. \(c_1\) is the velocity at which points 1, 2, 3 and 4 travel as viewed from a stationary frame of reference.](image-url)
It is further assumed that the region bounded by the control volume (marked by the dashed line in Fig. 5) travels to the left at a constant speed \( c_2 \) without any changes in the shape of the interfaces. It is reasonable to assume that this simplification is valid for the cases in which the time scale of the intermediate stage is much larger than the one required for the three-layer system in the central region to reach a stage in which the pressure gradient is balanced by advection. It is difficult to determine the time required for such a balance, but the assumption is later justified by laboratory experiments. It should be noted that with the two assumptions discussed above, the only domain in which the interface does change its shape with time is the region between cross sections (a) and (b) in Fig. 5.

We take a coordinate system which travels with the origin at a speed \( c_2 \). In this frame of reference, the flow in the control volume appears steady. The velocities upstream (\( u_2 \) and \( u_3 \)) and downstream (\( u_3 \) and \( u_4 \)) are taken to be independent of depth, and on the boundaries of the control volume the pressure is assumed to be hydrostatic. Following von Kármán's (1940) study on gravity currents, we assume that the velocity of the intermediate layer at the origin vanishes due to the discontinuity of the interface slope. With this assumption application of the Bernoulli principle to the upper boundary of the intermediate layer yields

\[
\begin{align*}
    u_3^2 &= 2g^*\eta, \\
    P_A &= -\rho u_1^{3/2},
\end{align*}
\]

where \( u_2 \) is the velocity downstream, \( g^* \) the "reduced gravity" given by \( g^* = g(\Delta \rho / \rho) \), \( \eta \) the depth of the upper and lower layers, \( P_A \) the pressure under the lid upstream at point (A) and \( u_1 \) is the velocity upstream.

Application of the Bernoulli equation to the lower layer along the bottom gives

\[
(\rho + \Delta \rho)u_3^{3/2} + P_A + \Delta \rho \xi g = (\rho + \Delta \rho)u_4^{3/2},
\]

where \( \xi \) is the upstream depth of the lower layer and \( u_3 \) and \( u_4 \) are the lower layer velocities upstream and downstream, respectively. By applying the Boussinesq approximation and considering (3), the energy equation (4) becomes

\[
u_3^2 - u_1^2 + 2g^*\xi - u_4^2 = 0.
\]

The momentum equation for each layer is

\[
F_s = \int \rho \mathbf{v} \cdot d\mathbf{A},
\]

where \( F_s \) is the force on the control surface (cs) which bounds each layer, and \( \mathbf{v} \) and \( d\mathbf{A} \) are the velocity and surface vectors, respectively. Application of (6) to the upper, intermediate and lower layer gives

\[
F_1 = (\rho - \Delta \rho)g \eta^{3/2},
\]

\[
\rho(H - \xi)u_3^2 + P_A(H - \xi) + F_2
\]

\[
= F_1 + \rho(H - 2\eta)u_3^2 + \frac{1}{2} \Delta \rho g[2H(\xi - \eta) - \xi^2]
\]

\[
+ \frac{1}{2} \Delta \rho g(4\eta^2 - 2H\eta),
\]

\[
\rho + \Delta \rho)\xi u_3^2 + P_A \xi
\]

\[
= F_2 + (\rho + \Delta \rho)\eta u_4^2 + \frac{1}{2} \Delta \rho g\eta^2 + \xi^2
\]

\[
\times [2H(\eta - \xi) + \xi^2 - \eta^2] - \frac{1}{2} \Delta \rho g(\eta^2 + \xi^2),
\]

where \( F_1 \) and \( F_2 \) are the unknown forces which act across the interfaces and it has been assumed that the pressure on the boundaries of the control volume (shown by the dashed lines in Fig. 5) is hydrostatic. By eliminating the forces \( F_1 \) and \( F_2 \) from (7), (8) and (9), applying the Boussinesq approximation and considering (3), we obtain

\[
u_3^2(\frac{1}{2}H - \xi) + u_3^2\xi + g'(\frac{1}{2}\xi^2 + H\eta - \eta^2)
\]

\[
- u_3^2(H - 2\eta) - u_4^2\eta = 0.
\]

The continuity equations for the intermediate and lower layers are

\[
u_1(H - \xi) = (H - 2\eta)u_2,
\]

\[
u_3 = u_4\eta.
\]
Eqs. (2), (5), (10), (11) and (12) contain six unknowns and an additional condition is required in order to solve the set. Note that in a stationary frame of reference there can be no flow in the intermediate layer in the center of the tank. Therefore, the speed at which our coordinate system is traveling (c_2) is simply u_2. In view of the symmetry of the problem, this condition yields

\[ u_2 = 2u_1. \]  \hfill (13)

Substitution of (2), (11), (12) and (13) into (5) gives

\[ \phi_1(\xi^*, \eta^*) = 4(\eta^*)^3(\xi^*)^3 + \xi^* - 4\eta^* - \eta^*(1 - 2\eta^*)^2/(1 - \xi^*)^2 = 0, \]  \hfill (14)

where the nondimensional variables \( \xi^* \) and \( \eta^* \) are defined by

\[ \xi^* = \xi/H, \quad \eta^* = \eta/H. \]  \hfill (15)

Similarly, substitution of (2), (11), (12) and (13) into (10) yields

\[ \phi_2(\xi^*, \eta^*) = \eta^*(1 - 2\xi^*)^2/2 - \eta^* - 5(\eta^*)^2 = 0. \]  \hfill (16)

Eq. (16) shows that for each \( \eta^* \) four roots of \( \xi^* \) can be found, while (14) shows that three roots of \( \eta^* \) exist for each \( \xi^* \). Therefore, one concludes that the set (14) and (16) may have as many as 12 separate pairs of roots.

Eqs. (14) and (16) were solved numerically by finding the intersections of \( \phi_1(\xi^*, \eta^*) = 0 \) and \( \phi_2(\xi^*, \eta^*) = 0 \) in the \( (\xi^*, \eta^*) \) plane. In the physical range \( 0 \leq \xi^* \leq 1.00 \) and \( 0 \leq \eta^* \leq 0.500 \), only two pairs of roots were found

\[ \xi_{1}^* \approx 0.403, \quad \eta_{1}^* \approx 0.355, \]  \hfill (17)

\[ \xi_{2}^* \approx 0.674, \quad \eta_{2}^* \approx 0.060. \]  \hfill (18)

The functions \( \phi_1(\xi^*, \eta^*) \) and \( \phi_2(\xi^*, \eta^*) \) both equal zero and the corresponding roots are shown schematically in Fig. 6 which indicates that both roots appear to be stable.

The first pair (17) corresponds to the situation where the speed \( c_2 \) of the upper layer (point 2, Fig. 5) is 0.843 \((g'H)^{1/2}\), while the speed \( c_1 \) of the inter-

![Fig. 6. Schematic diagram of the functions \( \phi_1(\xi^*, \eta^*) = 0 \) and \( \phi_2(\xi^*, \eta^*) = 0 \) and the corresponding pairs of roots.](image)

mediate layer (point 1) is only 0.500 \((g'H)^{1/2}\). Note that \( c_1 \) and \( c_2 \) are the velocities as viewed from a stationary frame of references. The relative speeds show that the intermediate water sinks rapidly from the surface. This sinking is completed (that is to say, point 2 catches up with point 1) at a distance of 3.46L (where L is the initial length of the intermediate water) from the center of the tank. The time \( t \) required for the completion of the sinking is

\[ t = 6.10L(g'H)^{-1/2}, \]  \hfill (19)

where \( t \) includes the time required for the completion of both the first and the intermediate stages. Note that as mentioned earlier the interface between cross sections (a) and (b) in Fig. 5 changes its shape and position with time. Therefore, the volume of the upper and the lower layers in this domain also varies with time and the flux which enters each layer does not equal the flux that leaves.

The second pair of roots (18) corresponds to the situation shown in Fig. 7. Point 2 travels to the left at a speed of 0.346 \((g'H)^{1/2}\) much slower than point 1, which travels at 0.500 \((g'H)^{1/2}\). It is difficult to determine whether or not the transition between

![Fig. 7. Intermediate intrusion stage corresponding to the second pair of roots.](image)

All the velocities are as viewed from a stationary frame of reference.
the central region and the far field contains a shock as shown by the dashed line in Fig. 7. However, it does not seem worthwhile to investigate this problem, since it seems that the complete solution which corresponds to the second pair of roots would not exist under normal conditions. This is justified by the following reasoning:

1) The transition from the first to the intermediate stage requires that the depth $\eta^*$ decrease from 0.500 to its value in the intermediate stage. Since both roots appear to be stable it is likely that $\eta^*$ will decrease to 0.355 (corresponding to the first pair of roots) without further decreasing to 0.060.

2) The first pair of roots leads to a final stage where the intermediate water is concentrated in mid-depth and the flow is stable (see Fig. 8). The second pair of roots leads to a situation where the intermediate water concentrates on the surface—a situation which probably cannot be sustained.

While these arguments do not prove the uniqueness, the laboratory experiments support the choice of the first root.

c. The final stage

During the final stage (Fig. 8) the velocity field is time dependent since the layers change their depth with time. The detailed solution for this case is beyond the scope of this study. However, we can say that since at $t \to \infty$ the intermediate layer vanishes, the velocity of points 1, 2, 3 and 4 at this time is simply $(2g'H)^{1/2}/2$.

4. The total solution

Thus far, the analysis has been confined to the upper portion. The total solution consists of a combination of the solutions to the upper and lower portions. We will show below that the solution to the lower portion has the same characteristics as the solution to the upper portion.

By considering the densities shown in Fig. 3b one can show that

$$g'H = g^* \hat{H},$$

where $g'$ and $g^*$ are the reduced gravities for the upper and lower portions evaluated by considering the density difference between two neighboring layers. Eq. (20) shows that the velocities in the two portions are identical. Therefore, application of the Bernoulli principle to the boundary between the two portions shows that the matching condition (continuity of pressure) is satisfied; that is, the solution to the total problem during all the intrusion stages is simply the combination of the solutions to each portion. Fig. 9 shows the total solution for the final stage. It is interesting to note that in the absence of intermediate waters, the total problem resembles the cases considered by Hachey (1934) and Long (1977).

Fig. 8. Final intrusion stage for the three-fluid intrusion problem. The intermediate density fluid which sunk from the surface and rose from the bottom is concentrated in mid-depth.

Fig. 9. The total solution—final intrusion stage. This predicted density structure should be compared with a typical observed structure shown in Fig. 1. Note that the thick "nose" on the right hand side results from the approximation of the vertical density field by two layers. In reality, it is expected that the nose will be much thinner (see, e.g., Manins, 1976).
Before concluding our theoretical analysis, it is appropriate to examine the possible role of dissipative mechanisms. We shall first consider the possible dissipation effects on the first intrusion stage (Fig. 4). It is recalled that during this stage, the system behaves in the same manner as the mutual intrusion of two fluids, since the intrusion on each side does not feel the presence of the other, due to the distance between the two. Note that even if the initial density structure is not symmetrical (i.e., the density of the intermediate fluid is not equal to the average of the densities on the left and right), the first intrusion stage will still behave as a two-fluid intrusion. However, under such asymmetrical conditions, the speed of the mutual intrusion on the left field will differ from the speed of the mutual intrusion on the right. Whether the initial total field is symmetrical or not, the depth of each fluid at $x = \pm L/2$ (Fig. 4) must be one-half of the total depth, since the right and left fields are always symmetrical with respect to these axes. It will be shown later that this argument has been supported by laboratory experiments with asymmetrical fields.

Benjamin’s (1968) analysis of gravity currents shows that if dissipation is included and the shape of the interface is taken to be time-independent, then the depth of the invading fluid must occupy more than one-half the total depth. However, as mentioned above, the symmetry of the problem with respect to the axes $x = \pm L/2$ shows that at least at these locations, each fluid must occupy half of the total depth. Therefore, it is believed that Benjamin’s time-independent dissipation analysis is inapplicable to the mutual intrusion of two fluids and to our first intrusion stage. It appears that dissipation effects cannot be understood unless one considers a time-dependent problem which is beyond the scope of this study. However, as will be discussed in the next section, the laboratory experiments show that while dissipative mechanisms may slow down the flow, they probably do not alter the general behavior of the field during the first and intermediate stages (i.e., the generation of the front).

### 5. Laboratory experiments

Many laboratory experiments of density currents and of two bodies of water separated initially by a vertical barrier have been conducted (see, e.g., Keulegan, 1957; Middleton, 1966; Yih, 1947). However, to my knowledge, there have been no experiments on the mutual intrusion of three or more bodies of water.

#### a. Apparatus

The experimental apparatus consists of a plexiglass tank 12.5 cm wide, 228.6 cm long and 38.1 cm high. The tank could be separated into three or more sections by a number of gates. The depth of the water varied between 9 and 13 cm and a typical density difference was $10^{-2}$ g cm$^{-3}$.

The apparatus was designed such that frictional forces which arise due to the bottom, the free surface and vertical walls have a limited effect on the flow. This has been achieved by choosing the apparatus characteristics such that the Reynolds number (evaluated by considering the molecular viscosity) is of order several hundreds. However, the boundary layer thickness was small in comparison to the fluid width and depth only during the initial stages of intrusion. For example, Schlichting (1968) gives $5(\nu l/\mu)^{1/2}$ for the former, which for $l \approx 30$ cm, $\mu \approx 0.5$ cm s$^{-1}$ and $\nu = 10^{-2}$ cm$^2$ s$^{-1}$ is about 3.5 cm, a thickness comparable to the tank width and depth. Thus, the details of the intermediate and final stages will be influenced by side and bottom friction. It will be seen later that frictional effects are apparent in the photographs.

#### b. Methods of visualizations and measurements

The upper and lower depth $\eta$ and the velocities in the intermediate stage were measured by photographing the intrusion during a recorded period of time. For flow visualization, colored dyes were injected into the various fluids prior to the lifting of the gates. Preliminary experiments showed that the curvature of the interfaces extends for relatively long distances. The interface to the left of the origin, below point A in Fig. 5, had a tilt of several degrees and was not horizontal as assumed in the theoretical model. This probably resulted from the relatively short length of the apparatus and from frictional effects. The experimental results show that this small interface tilt did not have a significant influence on the flow. Although the flow was not exactly horizontal, the pressure field at the control surface was approximately hydrostatic as assumed in the theoretical model. However, the interface tilt made the measurement of $\xi$ impossible, due to the difficulty in determining the location at which it should be measured.

In contrast to the interface below point A, the interface to the right of the origin was horizontal in the center of the tank, due to the symmetry of the flow. Hence, $\eta$ was measured at the center of the tank. As judged from visual observations, the influence of the tank end walls was limited to a distance of $O(H)$. Measurements were performed only with symmetrical fields, which included one intermediate body of water with constant density, but qualitative experiments were also conducted with asymmetrical fields and with two, three and four intermediate bodies.

Due to observational and mechanical limitations, the distance between the gates was fixed and the
range of the other parameters (depth and density) was relatively small. The details of this range of parameters is given in Table 1. During each experiment, about 15 recorded photographs were taken. This enables one to obtain several measurements of the velocity and depth from each experiment.

c. Experimental results

A sequence of photographs of a typical experiment is shown in Fig. 10, which includes the central and the left portion of the tank. These photographs clearly verify the existence of the three stages of intrusion previously described. When two, three or four intermediate bodies of water with a density increase from right to left were present, the intrusion followed the general characteristics of the one intermediate water case. That is, waters with intermediate density ultimately concentrated at mid-depth; as the intrusion continues, the layers are arranged in such a way that the density increases downward as shown qualitatively in Fig. 11. Photographs of these experiments are not presented here due to the difficulty of distinguishing between the different bodies of water in a black and white print.

When the initial density field was asymmetrical, the mutual intrusion followed the general characteristics of the symmetrical field. That is, the fluid with intermediate density (unequal to the average of the densities on the right and left sides) sunk from the surface, rose from the bottom and ultimately concentrated at mid-depth. However, under such asymmetrical conditions, the sinking velocity differed from the rising speed and point 2 caught up with point 1 (Fig. 4) after or before point 3 caught up with 4, depending on the intermediate density. When the density difference between the intermediate fluid and the fluid on the right (Fig. 4) was larger than the density difference between the intermediate fluid and the fluid on the left, point 2 caught up with point 1 before point 3 caught up with 4. Similarly, when the density difference between the intermediate fluid and the fluid on the right was smaller than the density difference between the intermediate fluid and the fluid on the left, point 2 caught up with point 1 after point 3 caught up with 4. Although an attempt to theoretically analyze asymmetrical fields was not made, the reader can qualitatively verify, by following the three stages of intrusion discussed in Section 3, that the behavior described above is expected.

We shall now discuss the experimental results of symmetrical fields with one intermediate body of water. A comparison of the velocity fields \( c_1; c_2 \) and the upper and lower layer depth \( \eta \) to the values predicted by the theory is shown in Fig. 12. For each intruding fluid and value of \( (g'H)^{1/2} \), there are two values of \( \eta/H \). One of these corresponds to the intruding fluid near the free surface and the other to the flow near the bottom of the tank. As expected, the intrusion near the bottom was slower than the one near the free surface due to frictional effects. Similarly, the two values of \( \eta/H \) correspond to the intrusion near the surface and bottom. The lower value corresponds to the depth near the bottom. \( \eta^* \) was found to deviate by 10–15% from the theoretical frictionless prediction and the velocities were about 85–90% of the predicted values. These deviations are probably due to frictional effects which were discussed earlier. It should be noted that Yih (1947), in his two-fluid intrusion experiments, found the velocities to be about 92% of the theoretical prediction. This corresponds to a deviation which is comparable to the one found in our study.

Although a substantial number of symmetrical experiments with one intermediate fluid were performed, none of them displayed the flow pattern shown in Fig. 7. Hence, the experimental results support the theoretical rejection of the second pair of roots and thus resolve the non-uniqueness question.

The shape of the interface near the free surface (see Fig. 10) indicates that due to contamination (such as wax and dust), the free surface did not follow the internal motions. That is, small particles in the free surface remained almost at rest while particles floating below it were advected by the intruding current. The photographs show that the thickness of the boundary layer below the free surface was of order several millimeters.

### Table 1. Details of the laboratory experiments with one intermediate fluid.

<table>
<thead>
<tr>
<th>Run</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (cm)</td>
<td>9.1</td>
<td>9.6</td>
<td>10.1</td>
<td>10.6</td>
<td>11.1</td>
<td>11.6</td>
<td>12.1</td>
<td>12.6</td>
</tr>
<tr>
<td>( \Delta \eta / \rho ) ((\times 10^9))</td>
<td>0.524</td>
<td>0.562</td>
<td>0.604</td>
<td>0.650</td>
<td>0.692</td>
<td>0.730</td>
<td>0.771</td>
<td>0.813</td>
</tr>
</tbody>
</table>

6. Summary and Discussion

The theoretical predictions and the results of the laboratory experiments for the three-layer model can be summarized as follows:

1) When a mutual intrusion of three bodies of water with different densities is taking place, three major stages of intrusion exist. In the first stage, the intrusion behaves essentially as an intrusion of two fluids. After some time, the two-fluid intrusion ceases to exist and an intermediate stage is established. During this stage the fluid with intermediate density sinks from the surface and rises from the bottom. This motion results ultimately in a concentration of the fluid with intermediate density in mid-depth (the final stage).
2) The first, intermediate and final stages exist for the periods $0 \leq t \leq L(g'H)^{-1/2}$, $L(g'H)^{-1/2} < t \leq 6.10 L(g'H)^{-1/2}$ and $6.10 L(g'H)^{-1/2} < t < \infty$, respectively. The location at which the transition between the intermediate and the final stage is taking place is independent of both the stratification and depth and is given approximately by $3.46L$ from the center of the initial position of the intermediate fluid.

These results were extended qualitatively to the more general case of the mutual intrusion of an infinite number of fluids with a gradual stratification decrease corresponding to a gradually varying mixing mechanism. As a result of the intrusion, the horizontal density gradients at the surface, bottom and intermediate depth increase dramatically (see Figs. 13 and 9). Fig. 13 shows the formation of a density discontinuity a finite amount of time after
the mixing has been relaxed. This time interval is given by (19), which for $\Delta \rho/\rho = 2 \times 10^{-3}$, $L = 1 \text{ km}$ and $H = 50 \text{ m}$ (estuarine fronts) yields $\sim 1.5 \text{ h}$. The predicted density structure (Fig. 9) displays a similarity to observations of fronts on the continental shelf (James, 1977, Fig. 1; Simpson and Allen, 1978; Simpson, 1976; Schumacher et al., 1978), and to recent observations of fronts in the Haro Strait (mentioned earlier in Section 1). The model also predicts the correct location of the surface and bottom fronts and may explain the associated sharp density gradients.

Since our model does not include rotational effects, the qualitative agreement found between the model and the density structure of shelf fronts whose length scale is $L = \lambda'$ [where $\lambda'$ is the internal

Rossby deformation radius $(g'H)^{1/2}/f]$ requires a further investigation. We shall show below that this qualitative agreement is not surprising and that mutual intrusion may occur even on a large scale provided that the flow is not in an exact geostrophic balance.

During the mutual intrusion, the pressure gradient force, which was created by the vertical mixing, is balanced by advection. One can see from Fig. 3a that on a large scale, the pressure gradient force could be balanced by the Coriolis force. If such a balance exists, no intrusion is taking place. But, since a considerable time is required for a flow to reach a geostrophic balance [$O(f)^{-1}$, where $f$ is the Coriolis parameter] one expects that intrusion will take place at least until a geostrophic balance is reached. Under such conditions intrusion and sharpening of gradients may occur until a geostrophic balance is obtained. Consequently, the final intrusion stage may not be reached but the horizontal density gradient will be sharper than the initial gradient as shown by the dash-dotted line in Fig. 13. In addition, in many coastal regions, the flow can never reach a geostrophic balance due to limitations imposed by the coast. In such cases, one expects that the portion of the pressure gradient force which is not balanced by the Coriolis force may drive an intrusion of a similar nature to the one studied here. It is believed that this may explain the qualitative agreement that is found between the nonrotating theoretical and laboratory model and the observations of James, Simpson and Schumacher et al., which were conducted in areas where the length scale is considerably larger than the Rossby internal deformation radius.

The above analysis suggests that it is the ageostrophic motions which are causing the sharp density gradients. It is interesting to note that Hoskins and Bretherton (1972) who examined the formation of fronts in a horizontal deformation field in the atmosphere\(^3\) reached a similar conclusion. They suggest that when a horizontal deformation field is pres-

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\(^3\) Their work was later applied to an ocean with an initial deformation field by O'Brien et al., (1977).
ent, the small-scale ageostrophic motions embedded in the baroclinic flow lead to a rapid formation of a near density discontinuity.

As stated earlier, the predicted fronts are constantly moving; their motion is combined from the intrusion speed and advection due to local currents. For instance, consider a long and narrow estuary or strait whose cross section does not vary in the field. Under such conditions, the tidal advection is mainly along its axis (along lines of constant depth) and is perpendicular to the direction of the intrusion. Therefore, the surface and bottom fronts will advance along the estuary and into the region of shoaling (see Figs. 9 and 2). Hence, they will be destroyed by the tidal mixing which follows the tidal cycle that generated the fronts. However, new fronts will be generated by mutual intrusion and under such conditions during each tidal cycle one set of fronts is destroyed and a new set is generated. It should be kept in mind, however, that the above analysis is based upon a frictionless theory and in reality the fronts will decay also due to frictional processes.

Finally, it should be emphasized that although qualitative agreement between the model and shelf fronts was found, at the present stage of the theory, the model cannot be quantitatively applied to frontal structures where $L < \lambda'$. That is, Eq. (19) and the corresponding equations are invalid when the bottom length scale is of the same order as the internal Rossby deformation radius. Further theoretical studies which include rotational effects are needed before a comparison to shelf fronts can be made. A model which includes effects of rotation, time-dependent mixing (corresponding to the tidal cycle), friction and steep bottom slopes will probably describe the processes in a more realistic manner for both shelf and estuarine fronts.

The applicability of our model to estuarine fronts where $L < \lambda'$ could perhaps be tested in the field by tracing the “intermediate waters” during the generation of the front, and comparing their path to the one predicted by the theory and the laboratory experiments.

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REFERENCES


Long, R. R., 1972: Finite amplitude disturbances in the flow of inviscid rotating and stratified fluids over obstacles. An-