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The role of angular momentum in the splitting of isolated eddies

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ABSTRACT

The question of which oceanic eddies can split and break up is addressed with the aid of two simplified analytical models which rely on the conservation of integrated angular momentum. First, an inviscid barotropic model with an initial round vortex is considered. The conditions necessary for the breakup of the vortex without exchanging angular momentum with its environment are examined. A solution for the final state is obtained without solving for the highly nonlinear transient splitting process. *It is found that only cyclonic eddies meet the necessary condition for splitting—anticyclonic eddies can never split, no matter what their structure is.* The cyclones are subject to a critical intensity above which breaking is possible and below which splitting is impossible. Specifically, cyclones with relative vorticity higher than f (where f is the (uniform) Coriolis parameter) can split into 2 eddies, whereas cyclones with a vorticity higher than $f/3$ can split into 3 or 4 vortices. The peculiar asymmetry between anticyclones and cyclones is a result of the conservation of integrated angular momentum. This can be demonstrated by noting that during the splitting process, the newly formed eddies are pushed away from their original center of rotation acquiring planetary torque. Therefore, in order for splitting to occur, the torque of the parent eddy must be large enough to accommodate for this addition of planetary torque. It turns out that only cyclones, which typically have more absolute angular momentum than their anticyclonic counterparts (because they rotate in the same sense as the spin of the earth), have enough torque to allow splitting. The above analysis is also applied to the splitting of a fully nonlinear zero potential vorticity lens. As in the barotropic anticyclonic cases, splitting is strictly impossible because the parent eddy does not have enough angular momentum.

1. Introduction

The question of eddy splitting is an important one because of the associated transfer of energy, momentum and mass from large to small scale. Oceanic observations suggest that eddy fission is not at all a simple process. Despite close examinations of many anticyclonic rings, there have not been any observations of such rings that break up. On the other hand, there is some observational evidence that cold-core rings (Fig. 1) do split up (see e.g., Cheney et al., 1976; Richardson et al., 1977; Ring Group, 1981; Hagan et al., 1978).

In view of these and the fact that some eddies are unstable (see, e.g., Saunders, 1973; Griffiths and Linden, 1981; Cushman-Roisin, 1986; Ripa, 1987), it is of interest to examine the fission process from a theoretical point of view. Using idealized models, we shall demonstrate analytically that, just as the oceanic observations suggest, anticyclonic rings can never split whereas cyclonic rings meet the necessary condition for breaking if their intensity is beyond a "critical" value. To do so we shall examine the conservation of angular momentum and the conditions required for the breakup of an idealized eddy to a set of stationary eddies adjacent to each other. We shall connect the final steady state to the original state without examining the highly nonlinear transient splitting phenomena. Such a connection can be done with the aid of

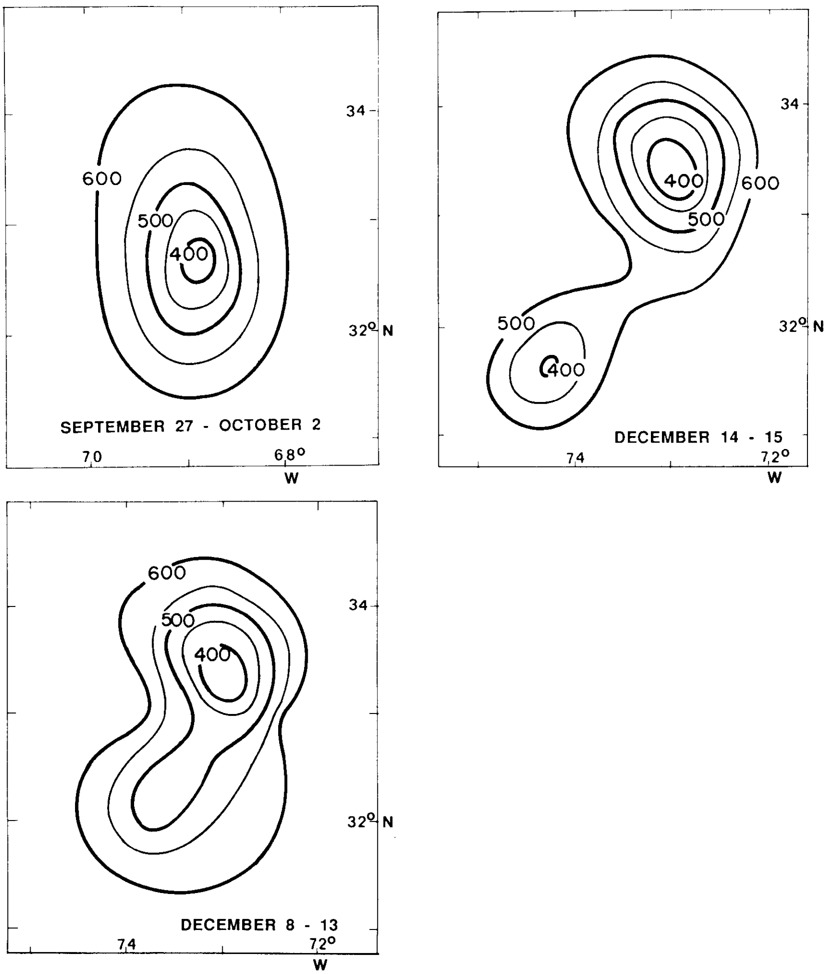

Fig. 1. A sequence showing the structure of a cold-core ring in the North Atlantic during September and December 1974. Depth (m) of the 15°C isothermal surface is based on ship surveys with 760 m XBT's (adapted from Cheney et al. 1976). Note that the cyclonic ring appears to be splitting into two off-spring.

conservation of torque, vorticity and mass. As in similar adjustment problems, some energy must be allowed to radiate away via long gravity waves.

So far, there have been no direct analytical attempts to study the fission of eddies. A somewhat related problem has been addressed by Thompson and Young (1989) who attempted to compute the breakup of a straight strip of current into a chain of vortices. However, they did not apply the conservation of angular momentum and, hence, could not close the problem; they derived an "upper bound" for the resulting eddies' size. The same problem was studied numerically by Salmon (1983) and experimentally by Griffiths et al. (1982).

We shall begin our analysis with an examination of the conservation of integrated angular momentum for an anomalous patch of fluid bounded by a vortex sheet (Section 2). With the aid of the integrated torque relationship (derived in Section 2) we shall first look at a simplified barotropic vortex (Section 3) and then proceed and address the fission of zero potential vorticity lenses (Section 4). The results are discussed in Section 5 and summarized in Section 6.

2. Torque

This section is devoted to a detailed examination of the conservation of integrated angular momentum in a rotating fluid. For the conservation of angular momentum in nonrotating systems ($f = 0$) the reader is referred to Csanady (1964, pages 88 and 112).

The conservation of torque for the special case of a rotating layer whose depth vanishes along its boundary was first discussed by Ball (1963). His results are extended here to the more general case of a region with anomalous vorticity bounded by a vortex sheet along which the depth is not necessarily zero nor is it necessarily constant (Fig. 2). Consider a patch of fluid bounded by a vortex sheet obeying the usual shallow water equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0, \tag{2.2}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \tag{2.3}$$

where the notation is conventional (i.e., f is the Coriolis parameter, u and v are the horizontal velocity components in the x and y directions, $h\,(= H + \eta)$ is the total depth and g is the gravitational acceleration). As usual, for a "reduced gravity" model g should be replaced by g' (where $g = \Delta\rho/\rho$, with $\Delta\rho$ being the density difference).

Multiplication of (2.1) by hy, (2.2) by hx, subtraction of the first resulting equation from the second, and consideration of the continuity equation gives,

$$\frac{\partial}{\partial t}[h(xv - yu)] + \frac{\partial}{\partial x}(huvx - hu^2 y) + \frac{\partial}{\partial y}(hxv^2 - huvy) + fh(xu + yv) - \frac{gy}{2}\frac{\partial}{\partial x}(h^2) + \frac{gx}{2}\frac{\partial}{\partial y}(h^2) = 0. \tag{2.4}$$

We now note that a multiplication of (2.3) by $(x^2 + y^2)/2$ gives,

$$\frac{\partial}{\partial t}[h(x^2 + y^2)/2] + \frac{\partial}{\partial x}[hu(x^2 + y^2)/2] + \frac{\partial}{\partial y}[hv(x^2 + y^2)/2] = hux + hvy,$$

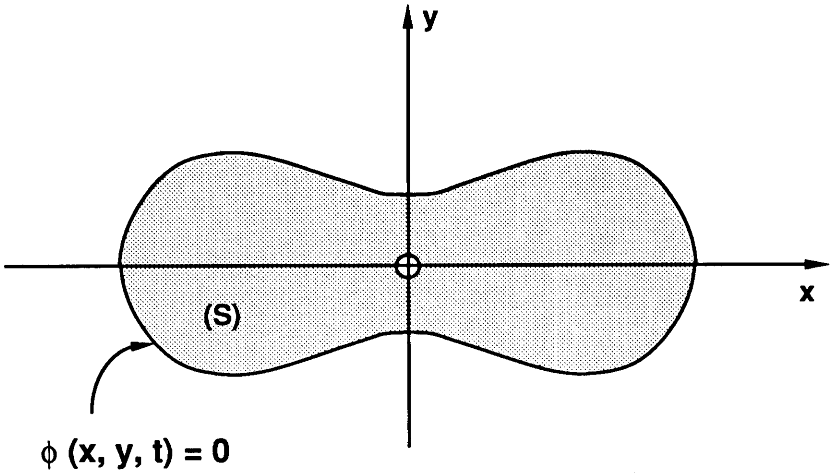

Fig. 2. Schematic diagram of a patch of barotropic fluid bounded by a vortex sheet. The vorticity in the patch is different from that outside the line $\phi(x, y, t) = 0$.

which upon substitution into (2.4) yields,

$$\frac{\partial}{\partial t}[h[xv - yu + (x^2 + y^2)/2]]$$
$$+ \frac{\partial}{\partial x}[hu[xv - yu + f(x^2 + y^2)/2]]$$
$$+ \frac{\partial}{\partial y}[hv[vx - yu + f(x^2 + y^2)/2]]$$
$$- \frac{g}{2}\frac{\partial}{\partial x}(yh^2) + \frac{g}{2}\frac{\partial}{\partial y}(xh^2) = 0. \qquad (2.5)$$

Using (2.3) to eliminate $\partial h/\partial t$ from the first term, and employing the definition of the total derivative (i.e., $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y$) to express the term

$$\frac{\partial}{\partial t}[xv - yu + f(x^2 + y^2)/2],$$

eq. (2.5) can be written as

$$h\frac{\mathrm{D}}{\mathrm{D}t}[xv - yu + f(x^2 + y^2)/2]$$
$$- \frac{g}{2}\frac{\partial}{\partial x}(yh^2) + \frac{g}{2}\frac{\partial}{\partial y}(xh^2) = 0. \qquad (2.5a)$$

Integration of (2.5a) over the patch (S) gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S h[xv - yu + f(x^2 + y^2)/2]\,\mathrm{d}x\,\mathrm{d}y$$
$$- \iint_S [xv - yu + f(x^2 + y^2)/2]\frac{\mathrm{D}}{\mathrm{D}t}(h\,\mathrm{d}S)$$
$$- \frac{g}{2}\oint [yh^2\,\mathrm{d}y + xh^2\,\mathrm{d}x] = 0, \qquad (2.6)$$

where Stokes theorem for the conversion of surface to line integral has been used and the arrowed circle indicates counterclockwise integration. A proper interpretation of $\mathrm{d}S$ shows that the second surface integral vanishes (see e.g., Ball, 1963, p. 242) so that we get,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\iint_S h[(xv - yu) + f(x^2 + y^2)/2]\,\mathrm{d}x\,\mathrm{d}y - \frac{g}{2}\underbrace{\oint_{\phi=0}(yh^2\,\mathrm{d}y + xh^2\,\mathrm{d}x)}_{\mathrm{I}} = 0} \qquad (2.7)$$

which is our desired relationship for the conservation of integrated angular momentum.

It is easy to see that in Ball's (parabolic) case, the last line integral vanishes at all times because $h = 0$ along the boundary. By contrast, in the more general barotropic or baroclinic "reduced gravity" cases, h is not zero along the boundary nor is it necessarily constant along the edge so that it is not a priori obvious what the value of the line integral is. Note that the integral represents the torque exerted on the patch by the surrounding fluid. Also, note that it is possible to illustrate that for features symmetrical with respect to x and y, the line integral is identically zero. In fact, the integral will not vanish only when the boundary has some sort of an asymmetrical "turbine-like" structure.

For simplicity the line integral will be taken to be zero which is equivalent to the statement that no net torque is exerted on the patch by the exterior fluid. It is expected that violent splitting processes may involve asymmetrical turbine-like features for which the integral may not vanish, but it is of interest to examine "gentle" and slow fission processes that do not involve exchange of angular momentum with their environment. Hence, we shall take the integrated angular momentum (in polar coordinates) to be,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\iint_S h\left(\frac{fr^2}{2} + rv_\theta\right)r\,\mathrm{d}r\,\mathrm{d}\theta = 0.} \qquad (2.8)$$

Three comments should be made with regard to (2.7) and (2.8). First, note that Saunders' (1973) splitting experiments clearly support the "non-turbine" symmetry concept (see (his) Fig. 2). Second, note that the integrated pressure torque always vanishes if one assumes that the speeds outside the splitting eddy are negligible. This is the case because, for an exterior with negligible velocities, the depth along the boundary $\phi(x, y, t) = 0$ is constant (due to the continuation of pressure) so that I reduces to,

$$\frac{gh^2}{2}\oint y\,\mathrm{d}y + x\,\mathrm{d}x,$$

which is identically zero for any closed contour.

Even though it may not be necessary for the conservation of integrated torque, we shall restrict ourselves to barotropic eddies of O(10 km), much smaller than the typical barotropic Rossby radius O(1000 km). Under such conditions, the flux of the outside fluid (displaced by the vortex sheet during the splitting) is distributed over a rather large area resulting in negligible speeds. This neglect of speeds (but not necessarily the transports) outside the vortex is not new. It is essentially identical to the negligible exterior flow assumption made in a similar situation involving eddies and vortex sheets (Nof, 1988a, b). Also, since the boundary is slippery, the condition of weak flow outside is equivalent to the neglect of motion in a very deep lower layer of a two-layer model.

Third, it should be pointed out that, in view of (2.8), the integrated angular momentum may also independently vanish for the exterior because I goes to zero when the integration boundary is at infinity and the velocities decay rapidly enough as $x, y \rightarrow \infty$. Since there are, obviously, changes in the value of the integral $\frac{1}{2} \iint hrv_\theta r \, dr \, d\theta$ during the splitting process (due to the changes in the position of the fluid), it follows that the exterior integral of the relative angular momentum $\iint hrv_\theta r \, dr \, d\theta$ may not vanish even though the velocities are negligible. This non-vanishing of the integral does not violate our assumptions: it is consistent with the fact that the length scale outside the eddy is very large compared to the eddy length scale so that neither the volume flux nor the integrated torque vanish even though the velocities are negligible.

3. A single-layer model

Although the angular momentum principle derived in the previous section is applicable to many systems, we shall, for simplicity, consider first a single layer barotropic model. Consider then the single layer model shown in Fig. 3. The initial barotropic eddy consists of a vorticity patch ($v_\theta = \alpha r$, where v_θ is the initial tangential velocity, α is a constant, and r is the radius) bounded by a vortex sheet beyond which the ocean is stagnant. The relative vorticity (2α) can be either positive or negative and the velocity is discontinuous along the edge—a condition that is not uncommon in inviscid flows (see, e.g., Nof, 1988a, b). It is assumed that after a transient splitting process a final steady state is reached. The final state consists of 2, 3 or 4 stationary eddies that "kiss" each other as shown in Fig. 4.

As mentioned, we shall assume that the speeds (but not necessarily the transports) outside the splitting vortex are small and can be neglected. This is based on the fact that the vortex length scale [$\sim$O(10 km)] is much smaller than the barotropic deformation radius [$\sim$O(1000 km)] so that the outer fluid that is pushed aside by the moving boundary is distributed over a rather large area resulting in small speeds.

3.1. Constraints

The initial and final states are connected with the aid of the following principles.

(i) *Vorticity*: since the model is inviscid, potential vorticity is conserved so that (in polar coordinates),

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_{\theta i}) + f\right] \Big/ (H + \eta_i)$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_{\theta f}) + f\right] \Big/ (H + \eta_f) = f/H_p, \tag{3.1}$$

where H_p is the "potential vorticity depth," H the undisturbed depth, η the free surface displacement (measured positively upward), and, as before, r is the radius. The subscripts "i" and "f" denote the initial and final states. Note that, even though the free surface vertical displacement plays a crucial role in the dynamics (because it allows energy radiation via long surface gravity waves) it can be neglected in (3.1) since $(H - H_p)/H_p \sim O(1)$ whereas $(\eta/H) \sim O(10^{-3})$ for a barotropic ocean [$H \sim O(1000 \text{ m})$ and $\eta \sim O(1 \text{ m})$]. It will become clear later that, in fact, in all of our relationships and budgets η can be neglected.

In view of the above and the initial condition,

$$v_{\theta i} = \alpha r, \quad \text{where} \quad \alpha = \frac{f}{2}\left[\left(\frac{H}{H_p}\right) - 1\right], \tag{3.2a}$$

one finds that,

$$\boxed{v_{\theta f} = \alpha r + O\left(\alpha r \frac{\eta}{H}\right).} \tag{3.2b}$$

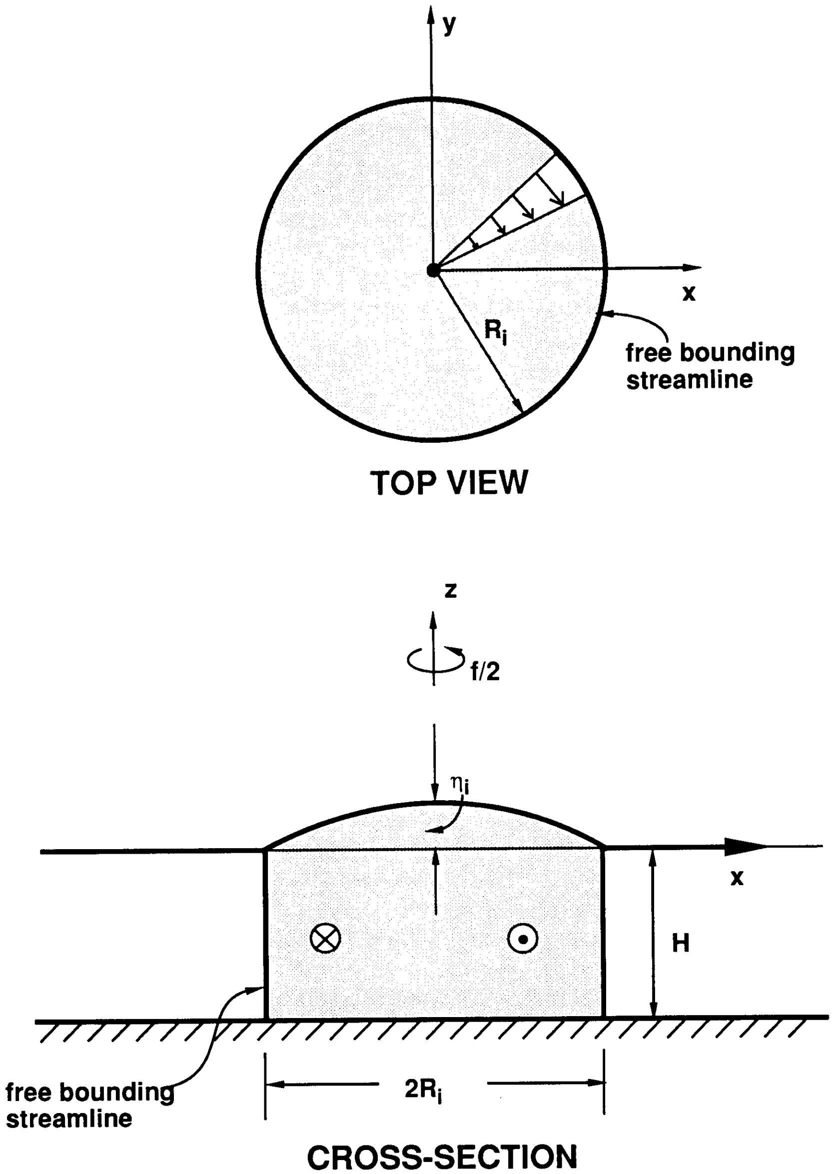

Fig. 3. Initial conditions for the barotropic model. The free surface vertical displacement η_i is measured upward from the undisturbed free surface. As in Nof (1988a), the velocity is discontinuous along the edge. The radius of the eddy is typically $\sim O(10\text{ km})$, much smaller than the barotropic deformation radius $\sim O(1000\text{ km})$. The subscript "i" indicates association with the "initial" state. Later on we shall use the subscript "f" to indicate the final state.

This essentially implies that the vorticity of the final off-spring is identical to that of the parent eddy (i.e., the velocity distributions are identical as well).

(ii) *Volume conservation*: this constraint can be written as,

$$\int\int_{S_i} (H+\eta_i)\, r\, \mathrm{d}r\, \mathrm{d}\theta = n \int\int_{S_f} (H+\eta_f)\, r\, \mathrm{d}r\, \mathrm{d}\theta, \quad (3.3a)$$

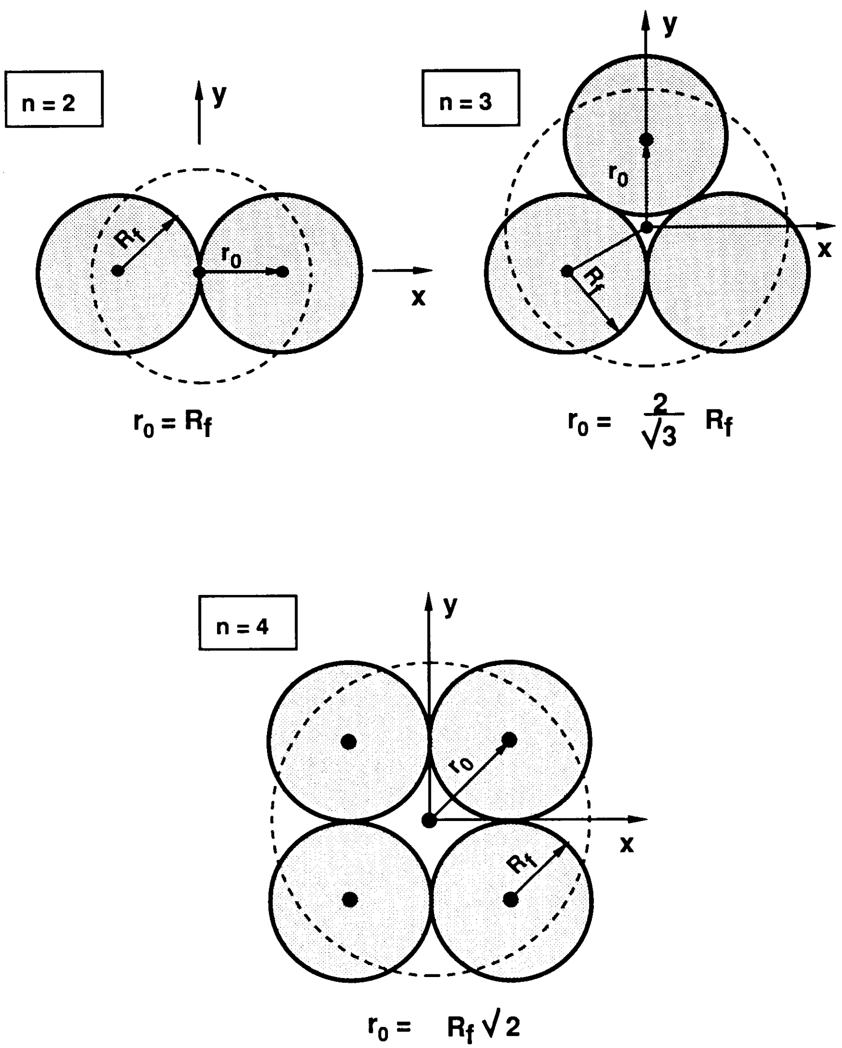

Fig. 4. The assumed final structure of the off-spring (i.e., the eddies resulting from the splitting). Three possible states are shown, n is the number of off-spring, r_0 is the distance from their center to the original center of rotation, and R_f is their radius. The dashed line shows the edge of the parent eddy.

where S denotes the area of the vortex and n is the number of off-spring. With $\eta/H \sim O(10^{-3})$, (3.3a) can be approximated by,

$$\boxed{R_i^2 = nR_f^2.} \tag{3.3b}$$

(iii) *Angular momentum*: following our earlier analysis in Section 2, we shall assume that the angular momentum (AM) of the patch is conserved:

$$\int\int_{S_i} [fr^2/2 + rv](H+\eta)\, r\, dr\, d\theta$$
$$= n \int\int_{S_f} [fr^2/2 + rv](H+\eta)\, r\, dr\, d\theta. \tag{3.4}$$

For a vortex (with radius R_f) whose center is situated a distance r_0 away from its original center of rotation, the integrated angular momentum is,

$$\mathrm{AM} = \int_0^{2\pi}\int_0^{R_f} [\tfrac{1}{2}f(r_0^2 + 2rr_0\cos\theta + r^2) + rv_\theta\cos\theta + rv_\theta](H+\eta)\,r\,\mathrm{d}r\,\mathrm{d}\theta, \qquad (3.5)$$

where r and θ are now measured from the center of the eddy. Since we focus on radially symmetric eddies ($\partial/\partial\theta = 0$), (3.5) takes the simpler form:

$$\mathrm{AM} = 2\pi\int_0^{R_f}(\tfrac{1}{2}fr_0^2 + fr^2/2 + rv_\theta)(H+\eta)\,r\,\mathrm{d}r. \qquad (3.5a)$$

This indicates that a vortex that is pushed a distance r_0 away from its original center of rotation acquires angular momentum due to the term $fr_0^2/2$. We shall see later that this fact is of crucial importance to the splitting process.

For our splitting vortex, (3.5a) and (3.4) give,

$$\boxed{\int_0^{R_f}(\tfrac{1}{2}fr^2 + rv_{\theta i})\,r\,\mathrm{d}r = n\int_0^{R_f}[\tfrac{1}{2}f(r_0^2 + r^2) + rv_{\theta f}]\,r\,\mathrm{d}r,} \qquad (3.6)$$

because, as before, $(\eta/H) \ll 1$. Note that relation (3.5) was also used by Cushman-Roisin (1989) who examined the merging of lenses (i.e., patches for which it is obvious that the angular momentum is conserved since $h = 0$ along the edge).

(iv) *Energy*: as mentioned, during the fission energy is radiated away via long gravity waves so that $E_i > E_f$, where E is the total available energy (i.e., the sum of the kinetic and available potential energy):

$$E = \iint_S [(H+\eta)v_\theta^2/2 + g\eta^2/2]\,\mathrm{d}s. \qquad (3.7)$$

Noting again that $(\eta/H) \sim 10^{-3}$ and $v_\theta^2 \sim O(g\eta)$ we find that (3.7) gives,

$$\Delta E = \frac{H}{2}\iint_{S_i} v_{\theta i}^2\,\mathrm{d}s - \frac{nH}{2}\iint_{S_f} v_{\theta f}^2\,\mathrm{d}s$$

or

$$\boxed{\Delta E = \pi H\left[\int_0^{R_i} v_{\theta i}^2\,r\,\mathrm{d}r - n\int_0^{R_f} v_{\theta f}^2\,r\,\mathrm{d}r\right].} \qquad (3.8)$$

which is our desired expression for the energy loss.

3.2. Connecting the final and the initial states

We shall first determine the conditions necessary for the splitting (of the parent eddy) into two off-spring ($n = 2$). For such conditions, (3.3b) gives $R_i = R_f\cdot\sqrt{2}$ which upon substitution into (3.6) (together with (3.2a)) gives,

$$\frac{1}{2} + \frac{\alpha}{f} = \left(\frac{r_0^2}{R_f^2}\right). \qquad (3.9)$$

Since the final eddies cannot overlap (see Fig. 3b), r_0 must be at least as large as R_f so that (3.9) yields,

$$\boxed{\alpha = f/2,} \quad n = 2, \qquad (3.10)$$

as the minimum value required for splitting (Fig. 5). Obviously, there can be no splitting for negative α (anticyclones) because for negative alpha (3.9) implies that $r_0 < R_f/\sqrt{2}$ which is impossible. One can see from (3.9) that cyclones with α greater than $f/2$ will split into a set of vortices that (instead of kissing each other) will be separated from each other (i.e., $r_0 > R_f$). Substitution of the appropriate quantities into (3.8) shows that the relative energy loss $[\Delta E/E_i = (1 - 1/n)]$ is exactly 50%.

The unusual cyclone–anticyclone asymmetry results from the condition that, as the off-spring are formed and pushed a distance r_0 away from their original center of rotation, they acquire planetary angular momentum $\frac{1}{2}\iint fr_0^2\,r\,\mathrm{d}r\,\mathrm{d}\theta$. Consequently, their final integrated torque is relatively large so that only parents whose angular momentum is large to begin with can produce them. Since cyclones rotate in the same sense as the spin of the earth, their absolute angular momentum is typically larger than that of their anticyclone counterparts, making them much more likely parents.

By repeating the above procedure for three off-spring ($n = 3$) one finds from the angular momentum constraint,

$$\left(1 + \frac{2\alpha}{f}\right) = \left(\frac{r_0}{R_f}\right)^2. \qquad (3.11)$$

In contrast to the two off-spring case where $r_0 \geqslant R_f$, we now have $r_0 \geqslant 2R_f/\sqrt{3}$ (see Fig. 3b) so

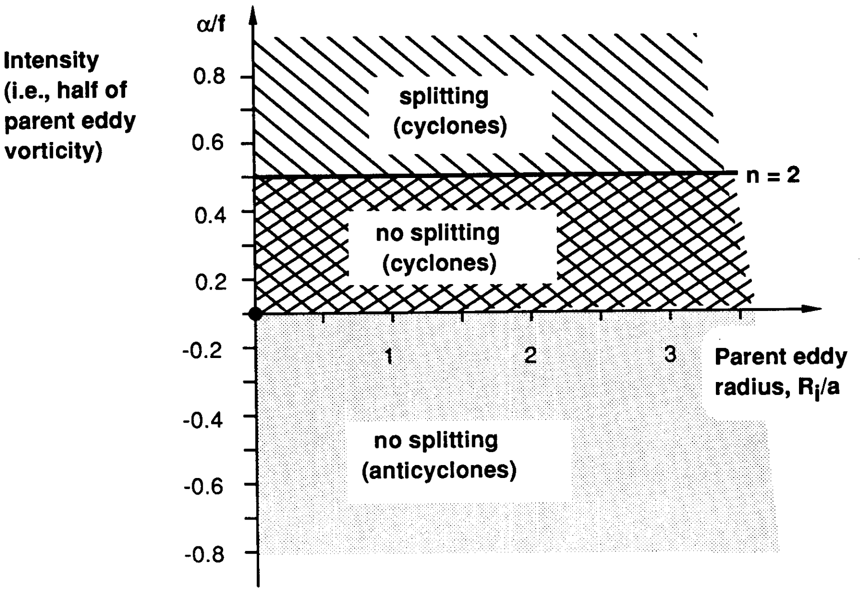

Fig. 5. Schematic diagram of the fission regimes for the barotropic model. For splitting into two off-spring ($n = 2$), it is necessary that the intensity of the parent vortex (α) be larger than $f/2$ (hatched area). Anticyclones (dotted area) correspond to negative α whereas cyclones (crossed area) are associated with positive α. For 3 and 4 off-spring the line separating splitting from no splitting is a horizontal line (parallel to the line corresponding to $n = 2$) intersecting the vertical axis at $f/6$ (not shown). The length scale "a" is a typical radius for the barotropic eddies (much smaller than the barotropic deformation radius R_d).

that the critical intensity is,

$$\boxed{\alpha = f/6,} \quad n = 3. \tag{3.12}$$

The energy loss is larger than that in the two off-spring case and amounts to about 67%.

For four off-spring ($n = 4$), one again finds that the critical intensity is,

$$\boxed{\alpha = f/6,} \quad n = 4, \tag{3.13}$$

so that the vorticity (2α) is $f/3$. Note that the energy loss is larger than those in the two and three off-spring cases and amounts to 75%.

4. Zero potential vorticity lenses cannot split and break up

Since many warm-core rings have a lens-like structure (Csanady, 1979; Flierl, 1979), it is of interest to examine their fission conditions. To do so we shall consider lenses overlying an infinitely deep passive lower layer with density ρ. For such nonlinear eddies the general form of the conservation of mass and angular momentum is very similar to that of the barotropic eddies,

$$\int_0^{R_i} h_i r\,dr = n \int_0^{R_f} h_f r\,dr, \tag{4.1}$$

$$\int_0^{R_i} h_i(\tfrac{1}{2}fr^2 + rv_{\theta i}) r\,dr = n \int_0^{R_f} h_f[\tfrac{1}{2}f(r_0^2 + r^2) + rv_{\theta f}] r\,dr, \tag{4.2}$$

where

$$v_{\theta i} = v_{\theta f} = -fr/2, \tag{4.3}$$

and $h_i = \hat{h} - f^2 r^2/8g'$ (here, $\hat{h}$ is the maximum lens' depth, $h(0)$, and g' is the "reduced gravity" $g\Delta\rho/\rho$ with $\Delta\rho$ being the density difference between the two layers). In view of (4.3), the left-hand side of (4.2) is identically zero so that for

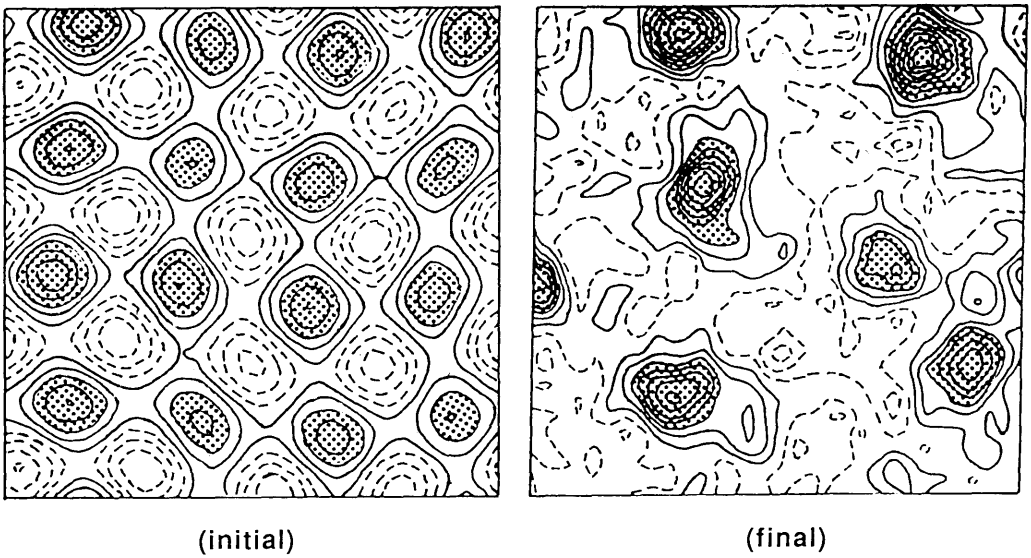

Fig. 6. Numerical simulations of a frontal geostrophic reduced gravity model showing the emergence of anticyclones from a field that initially contains an equal number of cyclones and anticyclones (adapted from Cushman-Roisin and Tang (1989)). The left panel shows the initial depth contours with relative amplitude ranging from -0.82 to 0.99; the right panel amplitudes range from -0.5 to 1.7. The time difference between the two panels is about 100 rotation periods. The size of each basin is $(16\pi R_d) \times (16\pi R_d)$ where, as before, R_d is the Rossby radius.

each off-spring it is required that,

$$fr_0^2 \int_0^{R_f} h_f \, r \, dr = 0, \tag{4.4}$$

a condition that obviously cannot be satisfied because it contradicts (4.1). The reason for the impossibility of splitting is that each of the (potentially) final vortices must have a total integrated angular momentum of $\frac{1}{2} fr_0^2 \iint h_f \, r \, dr \, d\theta$ due to the fact that it must have been pushed a distance of r_0 away from its original center of rotation. This means that the parent eddy must also have a positive angular momentum (relative to its own center). It is easy to see that this is not the case for a zero potential vorticity parent (because it has zero angular momentum) so that breaking up is impossible. It is of interest to note that an allowance for filamentation of the kind described in Cushman-Roisin (1989) does not resolve the difficulty in the splitting lenses model nor does it resolve the difficulty in the previously considered model. Also, an allowance for a steady orbital migration of the final lenses does not make splitting possible.

5. Discussion

Before comparing the results to actual oceanic observations, numerical simulations and laboratory experiments, it should be mentioned that due to our idealizations and approximations, a quantitative comparison is impossible. We shall shortly see, however, that a qualitative comparison can be made and that such a comparison produces favorable results.

5.1. Relationship to oceanic observations

As mentioned, despite close observations of many warm-core rings, fission of anticyclones has never been observed in the ocean. However, there have been two sets of observations that qualitatively describe the fission of cold-core rings. The first is reported in Cheney et al. (1976) and was shown in Fig. 1. The second involves rings Art and Al and their structure is described in Hagan et al. (1978), Richardson et al. (1977) and the Ring Group (1981). The fact that only cold-core rings have been observed to split is consistent with our analysis. Also, the relative abundance of mid-depth anticyclones in the ocean (McWilliams, 1985) is consistent with our results which state that cyclones might break up whereas anticyclones will remain coherent.

5.2. Relationship to numerical simulations

The peculiar cyclone–anticyclone asymmetry has been noticed—but was not fully explained—in at least 3 reported simulations. Williams and Yamagata (1984) observed that intermediate scale anticyclones last much longer than their cyclonic counterparts. They attributed the difference in life times to mean currents. Cushman-Roisin and Tang (1989) report that in generalized geostrophic turbulence only anticyclones ultimately emerge. As shown in Fig. 6, from an initial field of sixteen cyclones and sixteen anticyclones (left panel) only seven large anticyclones emerged (right panel). They propose that the effect may be, at least partly, due to the nonlinear β effect. Our analysis suggests, on the other hand, that such cyclone–anticyclone asymmetries may simply be a result of the fact that only cyclones meet the condition necessary for breaking up.

5.3. Relationship to laboratory experiments

Fission in the laboratory has been observed by Saunders (1973), Griffiths and Linden (1981), and Kostyanoy and Shapiro (1986). It is rather difficult to compare their results to our analysis because, in contrast to our model and the numerical simulations that involve only one active layer, laboratory eddies often have a very strong countervortex in the adjacent layer or some other form of coupling to the surrounding fluid. These features are a result of the way that anticyclones are usually produced in a rotating tank (see e.g., the collapsing cylinder aspect discussed in Nof and Simon (1987)). In view of these, it is sometimes impossible to say whether the splitting in the laboratory is the fission of a cyclone or the splitting of an anticyclone. For instance, it appears that in Saunders' experiments the cyclone on top was the most probable feature responsible for the breaking. This is supported by the experiments of Kostyanoy and Shapiro (1986) who found that only anticyclones with a cyclone on top broke up.

5.4. Relationship to other stability criteria

It is of interest to point out that a cyclonic–anticyclonic asymmetry is also found in the inertial instability case (Hess 1959, p. 306). This instability is also related to angular momentum; it is active whenever the total vorticity (curvature and shear) is negative implying that only anticyclones can be unstable. Our fission process and the inertial instability criteria are both asymmetrical but the asymmetry is not related because the two processes address different physical situations.

6. Summary

The main conclusion of this paper is that intense cyclonic eddies meet the necessary conditions for splitting (without exchanging angular momentum with their environment) whereas anticyclonic eddies do not. The detailed results of the study can be summarized as follows:

(i) An anticyclonic barotropic vortex that does not exchange angular momentum with its surrounding fluid cannot split into a set of two, three or four steady off-spring even if energy is added to it. Only cyclones meet the condition necessary for splitting without exchanging angular momentum. The cyclone-anticyclone asymmetry is due to the direction of the earth's rotation which opposes the rotation of anticyclones and adds to the rotation of cyclones. Specifically, as each of the (potentially) final off-spring is formed it must be pushed away from its original center of rotation so that it gains a significant amount of planetary torque (see equation (3.5a)). Since the integrated angular momentum is conserved, the potential parent eddy must also have a relatively large torque. It turns out that only cyclones can satisfy this condition because their total absolute torque is typically larger than that of their anticyclonic counterparts.

(ii) The necessary condition for the splitting of a barotropic cyclonic vortex that does not exchange angular momentum with its environment is that its vorticity be greater than a "critical value" (see Fig. 5). The critical splitting state corresponds to off-spring that are "kissing" each other (Fig. 4). When the vorticity is greater than the critical vorticity, the off-spring are separated from each other (i.e., their edges do not touch).

(iii) During the splitting process energy is radiated away via long gravity waves. In the barotropic case, 50% of the energy is lost when the cyclone splits into two off-spring, 67% when it splits into three and 75% when it splits into four off-spring.

(iv) Steep, zero potential vorticity lenses (i.e., anticyclones with a surfacing interface) cannot split and break up.

It is suggested that the cyclone–anticyclone asymmetry might explain the fact that warm-core rings have not been observed to split in the ocean whereas the splitting of cyclones has been qualitatively identified. In addition, the asymmetry is consistent with the fact that mid-depth anticyclones are abundant whereas mid-depth cyclones are much less common. Also, our results may provide an explanation for the observed asymmetry in numerical simulations of geostrophic turbulence. Such experiments display the emergence of anticyclones from a field that initially contained an equal number of cyclones and anticyclones (Fig. 6). Finally, it should be said that since we have merely addressed the conditions necessary (but, perhaps, not sufficient) for fission to occur, it would be useful to extend this study to a more complete transient splitting analysis. Extension to a general two-layer system would also be useful.

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