NOTES AND CORRESPONDENCE

The Breakup of Dense Filaments

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ABSTRACT

The breakup of a long strip of dense fluid flowing over a sloping bottom is examined with the aid of a nonlinear two-layer analytical model. The inviscid strip is bounded by the sloping bottom from below and an interface (that intersects the bottom along the two edges) from the top. The infinitely deep upper layer in which the filament is embedded contains a uniform flow and is taken to be passive. Such flows represent an idealization of currents that result from various outflows and deep water spreading.

It is shown analytically that a dense filament can break up to a discrete set of closely packed anticyclonic eddies (lenses) propagating steadily along the isobaths. The lenses are arranged in a zig-zag manner with the edges of each lens touching its neighboring lens. Such a pattern results from the fact that the eddies are too large to fit into the area freed by the straight filament so that they push each other to the sides during the breakup. The solution for this pack of eddies is computed without solving for the detailed breakup process. As in other adjustment problems, the final and initial states are connected via known conservation properties even though the problem is highly nonlinear. Specifically, conservation of potential vorticity, integrated angular momentum and mass are applied. These conservation laws illustrate that about 10% of the initial energy is radiated away (via long gravity waves) during the breakup.

The theory suggests that some of the actual filaments in the ocean, such as the Mediterranean outflow, may not consist of a single continuous flow but rather of a stream of closely packed lenses translating steadily along the bottom.

1. Introduction

Dense filaments have been observed in many parts of the deep ocean. They result from various outflows such as the Denmark (Worthington 1969; Mann 1969; Smith 1976), and the Mediterranean (Smith 1975; Ambar and Howe 1979a,b), or from intrusions of water from neighboring oceans (Weatherly and Kelly 1985). Theoretical studies have suggested that such currents are unstable (Griffiths et al. 1982, hereafter referred to as GKS) and, hence, might break up into a set of discrete eddies. The question of what is the outcome of such a process is the focus of this study. This question is of interest because it is associated with the transfer of energy, mass and momentum from large to small scale.

To examine the process under discussion we shall simplify the problem to that of a single current embedded in an infinitely deep upper layer and flowing along a sloping bottom (Fig. 1). The combined effect of a uniform current in the upper layer and a bottom induced speed \( g'S/f \) (where \( g' \) is the "reduced gravity" \( g\Delta p/\rho \), \( S \) the slope of the floor and \( f \) is the Coriolis parameter) permits the interface to strike the bottom along the two sides even though the absolute speed is one directional everywhere. We shall view the filament from a coordinates system moving at \( (U + g'S/f) \).

This eliminates the need to deal with the uniform flow above and the sloping bottom below (section 2). Since \( (U + g'S/f) \) is also the (along isobaths) translation speed of the final eddies (Fig. 2 and Nof 1983), the eddies appear to be stationary in our moving coordinates system. Namely, in the new coordinates system, the problem reduces to the breakup of a two-directional flow (on a flat bottom) to a set of stationary eddies (Fig. 3).

As mentioned, the stability of such filaments was theoretically examined by GKS; later on, Paldor and Killworth (1987, hereafter referred to as PK) extended GKS' theoretical results to two active layers and the entire breakup process was studied numerically by Salmon (1983). Treating the problem as an adjustment process (i.e., connecting the final and initial states without solving for the time-dependent process) and using energy conservation principles, Thompson and Young (1989, hereafter referred to as TY) found an expression for the size of the resulting eddies. Recognizing that energy can be radiated away via long gravity waves in the surrounding layer, they regard their predictions as an "upper bound." Since conservation of momentum is much more fundamental than conservation of energy (because it cannot be radiated away),

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we shall make an attempt to solve the problem using
conservation of angular momentum instead of con-
servation of energy. It is not a trivial matter at all to
apply the conservation of angular momentum to this
problem and, consequently, much of the analysis is
devoted to it (section 3).

We shall see that such an analysis indicates that the
final eddies cannot be arranged along a straight line
but rather must be staggered in a zig-zag manner. We
shall show that this is the case for both zero potential
vorticity filaments and finite potential vorticity flows
(section 4). It turns out that during the breakup about
10% of the energy is radiated away via long gravity
waves so that TY’s prediction for the eddy size is not
that far from the actual value (section 5). However,
their prediction for the geometry of the final arrange-
ment is quite far from the lenses’ zig-zag pattern.

2. Formulation

a. Governing equations

Consider again the layer-and-a-half model shown in
Fig. 1. A few comments should be made regarding this
figure. First, recall that the steady initial state consists
of a single layer flowing along the isobaths; it is embed-
ed in an infinitely deep upper layer containing a flow
with speed $U$. Second, note that the final state is as-
associated with a group of eddies steadily translating in
the same direction as the initial flow (Fig. 2). Third,
the manner in which the final eddies are geometrically
arranged is not known in advance but rather must be
determined as a part of the problem.

The “reduced gravity” shallow water equations gov-
erning any feature overlying a sloping bottom and sub-
merged in an infinitely deep upper layer flowing at $U$
are,

$$
\frac{\partial u_z}{\partial t} + u_z \frac{\partial u_z}{\partial x_z} + v_z \frac{\partial u_z}{\partial y_z} - f v_z = g_z \frac{\partial h}{\partial x_z} \quad (2.1)
$$

$$
\frac{\partial v_z}{\partial t} + u_z \frac{\partial v_z}{\partial x_z} + v_z \frac{\partial v_z}{\partial y_z} - f u_z = -g_z \frac{\partial h}{\partial y_z} \quad (2.2)
$$

$$
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_z} (h u_z) + \frac{\partial}{\partial y_z} (h v_z) = 0, \quad (2.3)
$$

where $h$ is the total depth, $g_z$ the “reduced gravity”
g $\Delta \rho / \rho$ (here, $g$ is the gravitational acceleration, $\Delta \rho$
the density difference between the layers and $\rho$ the density
of the filament), $u_z$ and $v_z$ are the horizontal velocity
components in the $x_z$ and $y_z$ direction, $t_z$ represents
time, $f$ is the Coriolis parameter and the subscript “$s$”
denotes association with a stationary coordinates sys-
tem. (The absence of a subscript will later denote a
traveling coordinates system.)

The relevant governing equations for the moving
coordinates system are obtained by applying the trans-
fornations $x_z \rightarrow x + [(g' S / f) + U] t_z$, $y_z \rightarrow y$
and $t_z \rightarrow t$ to (2.1–2.3). One finds,

$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g \frac{\partial h}{\partial x} \quad (2.4)
$$

$$
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g \frac{\partial h}{\partial y} \quad (2.5)
$$

FIG. 1. Schematic diagram of the idealized model under study.
The flow within the dense filament results from three components—
the uniform flow above ($U$), the slope induced speed $g' S / f$ [where
$g'$ is the “reduced gravity” $g \Delta \rho / \rho$, $S$ is the bottom slope (i.e., $S$
= tan $\alpha$) and $f$ is the Coriolis parameter] and the filament’s own
speed (Fig. 3). The subscript “$s$” indicates association with a sta-
tionary coordinates system.
\[ h = \frac{f^2}{2g'} (L^2 - y^2). \]  
\[ (2.10a) \]

The central depth \( H \) is now related to \( L \) via the simple relationship:
\[ L^2 = 2g'H/f^2. \]  
\[ (2.11) \]

c. The final state

This state consists of a group of steady, radially symmetric eddies. The flow is again governed by the potential vorticity and momentum equations which, in polar coordinates \((r, \theta)\), take the form
\[ \left[ \frac{1}{r} \frac{d}{dr} (rv_{\theta} + f) \right] h = \frac{f}{H_p} \]  
\[ (2.12) \]

and
\[ \frac{v_{\theta}^2}{r} + fr_{\theta} = \frac{g'}{r} \frac{dh}{dr}, \]  
\[ (2.13) \]

where \( v_{\theta} \) is the orbital speed. The set, (2.12)–(2.13), cannot be solved analytically in the most general case; namely, numerical integration is the only means that can provide solutions for \( (f/H_p) \neq 0 \).

As in the initial state, the zero potential vorticity limit is fairly simple. Taking \( H_p \rightarrow \infty \) in (2.12) shows that,
\[ v_{\theta} = -fr/2 \]  
\[ (2.14) \]

and
\[ h = \frac{f^2}{8g'} (R^2 - r^2). \]  
\[ (2.14a) \]

d. Connecting the initial and final states

Even though the general solution of the final eddies is now known, the eddies cannot be connected to the initial state without some additional information because their sizes, number and position are not known in advance. We shall see in the next section that con-

\[ \begin{array}{c}
\text{(p + } \Delta \text{p)} \\
\text{1/2} \\
\text{z} \\
\text{y} \\
\text{H} \\
\text{(p)} \\
\text{1/2} \\
\text{z} \\
\text{y} \\
\text{H} \\
\end{array} \]

FIG. 3. A cross section of the one dimensional (i.e., \( \partial/\partial x = 0 \)) filament in a coordinate system moving at \((g'S'f' + U)\). In this system, the bottom is taken to be flat and there is no uniform current above (see text).

\[ (\rho + \Delta \rho) \]

FIG. 2. A sketch of a group of isolated eddies, with different sizes and intensities but with equal densities, translating on a sloping bottom. The eddies are embedded in a uniform flow \((U)\) parallel to the isolobats. All eddies translate with the same speed \((C_s = g'S'f + U)\) at 90° to the right of the downhill slope (Nof 1983). Dashed lines indicate lines of equal depth; the depth increases with \( y \). The uniform translation speed results from the fact that the translation is independent of the properties of the eddy (adapted from Nof 1983).

\[ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = 0. \]  
\[ (2.6) \]

In this coordinates system both the initial state (Fig. 1) and the final state (Fig. 2) appear to be steady even though the breakup process itself is, obviously, time-dependent. Equations (2.4–2.6) imply that potential vorticity is conserved, i.e.,
\[ \frac{D}{Dt} \left( \frac{\partial v/\partial x - \partial u/\partial y + f}{h} \right) = 0. \]  
\[ (2.7) \]

b. The initial state

In the initial state \( v = 0 \) and \( \partial/\partial t = 0 \) so that the flow is geostrophic and the solution for \( h \) should satisfy \( h = 0 \) at \( y = \pm L \), where \( 2L \) is the width of the filament. For a given (uniform) potential vorticity \( f/H_p \) (where \( H_p \) is the “potential vorticity depth”), Eqs. (2.4–2.7) yield a second-order differential equation whose solution is
\[ h = H_p \left[ 1 - \frac{(e^{-y/L_R} + e^{y/L_R})}{(e^{-L/R_d} + e^{L/R_d})} \right], \]  
\[ (2.8) \]

where \( R_d = (g'H_p)^{1/2}/f \). In view of (2.8), the relationship between the central depth \( H \) [i.e., \( h(0) \)] and \( L \) is,
\[ \frac{H}{H_p} = 1 - 2/(e^{-L/R_d} + e^{L/R_d}). \]  
\[ (2.8a) \]

The solution for \( u \) is
\[ u = (g'H_p)^{1/2} \frac{-e^{-y/L_R} + e^{y/L_R}}{(e^{-L/R_d} + e^{L/R_d})}. \]  
\[ (2.9) \]

In the limit of zero potential vorticity \((H_p \rightarrow \infty)\), (2.8–2.9) give
\[ u = fy \]  
\[ (2.10) \]
servation of volume and angular momentum provides this additional necessary information.

3. Conservation of torque and momentum

a. Torque

The conservation of integrated angular momentum for the special case of a layer whose depth vanishes along its entire boundary was first discussed by Ball (1963). His results are extended here to the more general case of a region bounded by a line along which the depth is not necessarily zero nor is it necessarily constant. Consider the baroclinic system shown in Fig. 4 and suppose, temporarily, that the final eddies are arranged along a straight line as considered by Ty. Also, note that the patch of fluid originally bounded by the lines connecting ABCD obeys the shallow water equations, (2.4)–(2.6).

To derive the torque constraint we begin by multiplying (2.4) by \(hy\), (2.5) by \(hx\), subtracting the first resulting equation from the second, and considering the continuity equation which gives

\[
\frac{\partial}{\partial t} [h(xv - yu)] + \frac{\partial}{\partial x} (hu v - hu^2 y) + v \frac{\partial}{\partial y} (hx^2 - huv) + \frac{gy}{2} \frac{\partial}{\partial x} (h^2) - \frac{gx}{2} \frac{\partial}{\partial y} (h^2) = 0. \quad (3.1)
\]

We now note that a multiplication of the continuity equation by \((x^2 + y^2)/2\) gives

\[
\frac{\partial}{\partial t} [h(x^2 + y^2)/2] + \frac{\partial}{\partial x} [hu(x^2 + y^2)/2] + \frac{\partial}{\partial y} [hv(x^2 + y^2)/2] = hux + hvy \quad (3.2)
\]

which upon substitution into (3.1) yields,

\[
\frac{\partial}{\partial t} \{h(xv - yu + f(x^2 + y^2)/2)\}
\]

\[+ \frac{\partial}{\partial x} \{hu[vx - yu + f(x^2 + y^2)/2]\}\]

\[+ \frac{\partial}{\partial y} \{hv[vx - yu + f(x^2 + y^2)/2]\}\]

\[+ \frac{gy}{2} \frac{\partial}{\partial x} (y^2) - \frac{gx}{2} \frac{\partial}{\partial y} (x^2) = 0. \quad (3.3)
\]

Using the continuity equation (2.6) to eliminate \(\partial h/\partial t\) from the first term, and employing the definition of the total derivative [i.e., \(D/\partial t = (\partial/\partial t) + u(\partial/\partial x)\)]

\[+ v(\partial/\partial y)\] to express the term \((\partial/\partial t)[xv - yu + f(x^2 + y^2)/2]\), (3.3) can be written as

\[h \frac{D}{Dt} [xv - yu + f(x^2 + y^2)/2] + \frac{gy}{2} \frac{\partial}{\partial x} (y^2) - \frac{gx}{2} \frac{\partial}{\partial y} (x^2) = 0. \quad (3.4)
\]

Integration of (3.4) over an area \(S\) gives

\[
\frac{d}{dt} \int_S h[xv - yu + f(x^2 + y^2)/2] dxdy
\]

\[+ \frac{g'}{2} \int_S [y^2 dy + x^2 dx] = 0 \quad (3.5)
\]

where the Stokes' theorem for the conversion of surface to line integral has been used. A proper interpretation of \(dS\) shows that the second surface integral vanishes (e.g., see Ball 1963, p. 242, or Milne-Thompson 1960, p. 76) so that we get,
which is our desired relationship for the conservation of integrated angular momentum. Equation (3.6) states that the sum of (i) the change in time of the angular momentum integrated over an area $S$ and (ii) the pressure torque along the boundary of $S$ vanishes. In what follows we shall apply (3.6) to the segments from which each vortex originated (Fig. 4).

It is easy to see that in Ball’s parabolic bottom case and in Cushman-Roisin (1989), the last integral (I) vanishes at all times because $h = 0$ along the entire boundary. By contrast, in the more general case, $h$ is not necessarily zero along the entire boundary so that it is not a priori obvious what the value of the integrated pressure torque (I) is. It is possible to illustrate, however, that for symmetrical features of the kind shown in Fig. 4, the line integral is identically zero. To show this we first note that, since $h = 0$ along the sides (i.e., the lines corresponding to AD and BC), (3.6) reduces to

$$\frac{d}{dt} \int_S h(xv - yu + f(x^2 + y^2)/2) dx dy$$
$$+ \frac{g'}{2} \int_D h^2(xdx + ydy)$$
$$+ \frac{g'}{2} \int_B h^2(xdx + ydy) = 0$$  \hspace{1cm} (3.7)

where the last two integrals correspond to the position of the lines originating from DC and BA.

Second, we note that, because of the symmetry of the problem, the integral of any property along the line DC (and its subsequent position) is cancelled by its complimentary integral from B to A. In a way, this symmetry is equivalent to that common in problems with periodic boundary conditions where the properties along one boundary are identical to those in the other. Another way of thinking about the vanishing of the pressure torque (I) is in terms of angular momentum accumulation. Specifically, if the integral I is not zero, then it follows that there is an angular momentum flux from one segment (e.g., ABCD) to another implying infinite angular momentum at $x \rightarrow \pm \infty$ which is, of course, impossible. In view of the above considerations, we take the angular momentum (in polar coordinates) to be

$$\frac{d}{dt} \int_S h[fr^2/2 + rv_x] r dr d\theta = 0.$$  \hspace{1cm} (3.8)

b. Momentum

In addition to the integrated torque constraint mentioned above, there are two constraints resulting from conservation of integrated momentum.

The first is obtained by multiplying the $x$ momentum equation (2.4) by $h$, integration over the patch, consideration of a multiplication of the continuity equation by $y$, conversion of the surface integrals to line integrals and consideration of the symmetry conditions mentioned earlier. One ultimately finds

$$\frac{d}{dt} \int_S h(u - fy) dx dy = 0.$$  \hspace{1cm} (3.9)

Similarly, the second momentum constraint is obtained by using the $y$ momentum equation (2.5) and a multiplication of the continuity equation (2.6) by $x$

$$\frac{d}{dt} \int_S h(v + fx) dx dy = 0.$$  \hspace{1cm} (3.10)

It will become clear later that, for the problem at hand, (3.9) and (3.10) imply that the (initial) center of gravity of the patch must coincide with the (final) center of gravity of the eddies. This results from the fact that the integrals $\int \int hvdx dy$ are zero both initially and finally [because for the final state

$$\int \int hudx dy = -\int \int \frac{\partial \psi}{\partial y} dx dy = \oint \psi dx = 0$$

where the streamfunction $\psi$ is defined by $\partial \psi / \partial y = -uh$ and $\partial \psi / \partial x = vh$] so that in the final state,

$$\int \int hvdx dy = \int \int hxdx dy = 0.$$  \hspace{1cm} (3.11)

Namely, the system’s center of gravity remains unaltered.

4. Zero and finite potential vorticity filaments

Consider again the geometry shown in Fig. 4, i.e., the final eddies are arranged along the center of the initial filament as was the case in TY. Under this condition, the problem contains two unknowns—the final eddies radius $R$ and the length of the segment that initially breaks off, $2b$. By equating the volume of the segment that breaks off to the volume of the final eddy [using (2.10)–(2.11) and (2.14)] one obtains
\[ L^3 b = \frac{3\pi}{64} R^4. \]  \hspace{1cm} (4.1)

Similarly, application of the angular momentum constraint (3.8) to the initial and final states gives

\[ b = \left(\frac{3}{2}\right)^{1/2} L = 0.77 L \]  \hspace{1cm} (4.2)

so that the solution for \( R \) is

\[ R = \left[ \frac{64}{\pi (15)^{1/2}} \right]^{1/4} L = 1.51 L. \]  \hspace{1cm} (4.3)

This solution is unphysical because \( R > b \) which means that the eddies overlap (Fig. 5). The overlap results from the fact that the eddies are too large to fit into the area freed by the straight filament. We shall now show that this space difficulty is resolved if one allows the eddies to push each other to the sides, in such a manner that a final zig-zag configuration is established (Fig. 6). Under such conditions, the surface area occupied by the eddies is increased and, as we shall see, there will not be any overlap.

To show this, we note that the geometrical change does not alter the conservation of mass constraint (4.1), and that the vanishing of integral \( I \) (i.e., the boundary pressure torque) in (3.6) is still relevant even though the eddies are off center. We also note that, according to (3.8), the angular momentum (AM) of a vortex (with a radius \( R \)) situated a distance \( d \) away from the origin is

\[ \text{FIG. 5. Angular momentum conserving solution corresponding to the straight line pattern suggested by Thompson and Young (1989). This solution is unphysical because the eddies overlap.} \]

\[ \text{FIG. 6. The suggested zig-zag pattern of the final eddies; it is assumed here that the eddies are "kissing" each other. The configuration shown in Fig. 4 is impossible because the eddies turn out to be larger than the section from which they broke off (i.e., } R > b \text{) so that they overlap (Fig. 5). Consequently, they push each other to the sides and form the staggered geometry.} \]
AM = \int_0^{2\pi} \int_0^R \left[ \frac{1}{2} f (d^2 + 2rR \cos \theta + r^2) + rv \cos \theta + r v \right] h r d r d \theta, \quad (4.4)

where \( r \) and \( \theta \) are now measured from the center of the eddy. Since the eddies are radially symmetric, (4.4) takes the simpler form,

AM = 2\pi \int_0^R \left( \frac{1}{2} f d^2 + \frac{d^2}{2} + rv \right) h r d r, \quad (4.5)

which indicates that a vortex that is pushed a distance \( d \) away from its original position acquires planetary angular momentum due to the term \( f d^2/2 \).

Application of the torque constraint (4.5) and (3.8) to the configuration associated with the zig-zag geometry (Fig. 6) gives

\[ 5b^2 - 3L^2 = 15d^2. \quad (4.6) \]

Since the mass constraint (4.1) remains unaltered we have, as before,

\[ R^4 = \left( \frac{64}{3\pi} \right) L^3 b \quad (4.7) \]

and, due to the geometry, we also have

\[ R^2 = b^2 + d^2. \quad (4.8) \]

The system, (4.6)–(4.8), contains three unknowns \( R \), \( b \), and \( d \), all of which are functions of \( L \). One can easily form a single algebraic equation for \( b/L \),

\[ \frac{16}{9} \left( \frac{b}{L} \right)^4 - \frac{8}{15} \left( \frac{b}{L} \right)^2 - \frac{64}{3\pi} \left( \frac{b}{L} \right) + \frac{1}{25} = 0. \quad (4.9) \]

As in the straight line configuration, the system is subject to the condition of no eddies overlap. Namely, the distance \( b \) (Fig. 6) must be large enough to prevent eddy number 1 from overlapping eddy number 3 and so on; this condition is given by

\[ b \geq R/2. \quad (4.10) \]

Note that when the critical condition \( R/2 = b \) is met, eddy 4 "kisses" eddy 2 in addition to kissing eddy 1, which is shown in Fig. 6. Similarly, eddy 1 kisses eddy 3 in addition to the kissing of eddy 2. By taking (4.10) into account we find that (4.9) has only one physically relevant root:

\[ \left( \frac{b}{L} \right) = 1.63 \quad \text{(4.11a)} \]

with the aid of (4.7) and (4.6) this gives

\[ \left( \frac{R}{L} \right) = 1.82 \quad \text{and} \quad \left( \frac{d}{L} \right) = 0.82 \quad (4.11b) \]

which is our desired solution.

During the generation of eddies, energy is radiated away via long gravity waves in the surrounding layer. The ratio between the final \( (E_f) \) and the initial \( (E_i) \) energy is

\[ \frac{E_f}{E_i} = \frac{5\pi}{384} \left( \frac{R}{L} \right)^6 \left( \frac{b}{L} \right) = 0.91, \]

which implies that about 9% of the energy is lost during the breakup. Finally, it should be pointed out that our zig-zag geometry satisfies the integrated momentum constraints (3.11) because it is associated with an unaltered center of gravity.

As mentioned in section 2, there is no exact analytical solution for finite potential vorticity eddies. The numerically evaluated solutions for \( (R/L) \), (b/L) and (d/L) as a function of the potential vorticity are shown in Fig. 7. Note that in the limit of \( H_p \to \infty \) (zero potential vorticity) the values agree with those obtained earlier using the analytical method. As expected, there is no qualitative difference between the zero and the final potential vorticity cases. As in the zero potential vorticity case, the energy loss for finite potential vorticity is less than 10%.

5. Discussion

In order to compare our results to those of GKS, PK and TY it is necessary to introduce a new variable—

\[ \text{Fig. 7.} \quad \text{The variables (R/L), (b/L) and (d/L) as a function of the (finite) potential vorticity (f/H_p). The solution was obtained using a numerical integration for the eddy structure. In the limit H_p \to \infty (zero potential vorticity) the analytically computed values (4.11) are recovered.} \]
the relative width of the wedge that could conceptually produce the geostrophic current in the first place. This width was used extensively by GKS and TY and is defined by \( L_0 = w_0 f / (g' H_0)^{1/2} \), where \( w_0 \) and \( H_0 \) are the wedge’s width and depth. With the aid of this new variable, a comparison between our new theoretical predictions and those of the previous investigators is shown in Fig. 8. It is of interest to note that, for small \( L_0 \), the present theory, GKS, and PK all merge almost to one curve despite the fact that the theory approaches the problem from an entirely different point of view. For large \( L_0 \), the best fit is of PK which is the only analysis that is based on two active layers. In terms of wavelengths values and proximity to the data, our new model is not any better than the TY model or the GKS results. In fact, it appears that the TY results fit the data better than our new theory even though their model does not conserve angular momentum which is much more fundamental than energy because it cannot be radiated away. Since all the models under discussion are highly simplified, such a better fit should be regarded merely as a peculiarity.

The geometrical structure of the final eddies, as observed from the top by GKS, is shown in Fig. 9. The pattern suggests that the centers of the lenses are not situated along a circle [analogous to the straight line arrangement of TY] but rather in a wavy manner. Much of the wavy appearance involves the elongation of the eddies with their longer dimension oriented at an angle to the mid-line of the mean current. However, there also appears to be a similarity to our zig-zagged geometry (Fig. 6). Note that the recent numerical experiments of Pavia (1989) also show a similar pattern [see (his) Fig. 23].

It should be pointed out that in the limit of \( L \to \infty \) the distance between the two fronts becomes so large that one front does not feel the presence of the other and the initial geostrophic current should be stable (Paldor 1983). This implies that there should not be any breakup as \( L \to \infty \). Our solution does not reflect this property; it gives physically relevant values even when \( L \to \infty \) because it merely provides the connection between the initial and final states assuming that these two states are realizable.

![Figure 8](image-url)  
**Fig. 8.** The length of the (breaking) parallel geostrophic current \( 2b \) (nondimensionalized by the Rossby radius based on the central current depth) as a function of the relative current width \( \mathcal{L}_0 \) [here, \( \mathcal{L}_0 = w_0 f / (g' H_0)^{1/2} \), where \( w_0 \) and \( H_0 \) are the width and depth of the conceptual wedge that formed the geostrophic current in the first place]. Our results are the dashed-double-dotted curve (corresponding to a numerical solution of the finite potential vorticity lenses). The solid line is the result of the linear stability analysis of GKS where \( 2b \) is the wavelength of the most unstable mode. The dashed line is the energy conserving theory of TY. The dashed-dotted curve is PK’s result for the stability of a coupled front embedded in a layer that is five times deeper than the frontal layer. The two square-dotted points are from a numerical model by Salmon (1983). The data corresponds to the laboratory observations of GKS where \( 2b \) is now interpreted as the circumference of the annulus divided by the number of eddies observed. Points connected by a vertical line indicate the two extremes of wavelengths that increased with time. The solid dots correspond to \( w_0 = 2.0 \) cm and a wedge/tank depth ratio of about 0.2; circles correspond to \( w_0 = 3.5 \) or 2.9 cm and a depth ratio of about 0.2; crosses correspond to a depth ratio of 0.8.
7. Conclusion

The detailed results of our study can be summarized as follows:

The breakup of a dense filament submerged in a deep mean flow \( (U) \) over a sloping bottom (Fig. 1) can be viewed as the breakup of a flow over a flat bottom (Fig. 3). This is due to (i) the equations governing the former which transform to those governing the latter when the problem is viewed from a coordinates system moving at \( g' S/U \) (where \( g' \) is the "reduced gravity," \( S \) the bottom slope) and (ii) the fact that the resulting lenses migrate at \( g' S/U \) (see Fig. 2). The structure of the final eddies and their geometrical arrangement can be computed (without solving for the detailed time-dependent breakup process) by connecting the final and initial states using conservation of mass, potential vorticity and integrated angular momentum \( (3.6) \). Due to lack of space the final eddies cannot be situated along a straight line in the manner suggested by TY (Figs. 4 and 5). Instead, they push each other to the sides (during the breakup) and form a zig-zag geometry (Figs. 6 and 9). Our results suggest that actual filaments may not consist of a continuous density anomaly but rather of a closely packed group of lenses (Fig. 6) translating together along the isobaths. Since most data are processed through low pass filters (which eliminate high frequency motions), it may not be possible to identify the stream of lenses in conventional records.

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