Tellus (1991), *43A*, 64–67

Orbiting eddies

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(Manuscript received 6 February 1989; in final form 7 March 1990)

ABSTRACT

In this brief note, exact nonlinear solutions for lens-like eddies (i.e., blobs of oceanic water with anomalous density and anomalous vorticity) orbitting in an inertial circle are derived. The governing equations are nonlinear (i.e., ageostrophic) because the Rossby number is of order unity and the amplitude is of the same order as the maximum depth. Inviscid solutions for zero potential vorticity lenses are derived by transforming the equations of motion to a coordinate system traveling with the eddies along a circular path. It is shown that the lenses drift counter-clockwise at a tangential speed of fr_0, where f is the Coriolis parameter and r_0 is the radius of the orbit. The structure of the orbiting lens is identical to that of a stationary lens or a westward drifting lens, namely, the radius of the orbiting lens is $2\sqrt{2}$ times the Rossby radius and the swirl speed (i.e., the speed at which the fluid spins within the lens) is $-fr/2$ (where r is the radius). It is suggested that actual lenses in the ocean might very well display such an oscillatory behavior. However, high-frequency observations are necessary in order to identify these aspects.

1. Introduction

Isolated lenses containing anomalous water with an anticyclonic circulation are common in many parts of the ocean. In the upper ocean, such features are a result of meandering currents that pinch-off warm-core rings (see e.g., Brown, et al., 1983), at mid-depth they are typically associated with various outflows and mixing on continental shelves (e.g., McWilliams, 1985), and on the ocean floor they are probably a consequence of intrusions and mixing near the bottom (Armi and D'Asaro, 1980). The lenses are strongly nonlinear due to the high swirl speed and high amplitude.

The behavior of lenses on f and β planes has been studied extensively both analytically and numerically (see Flierl, 1987 for a review). However, the only exact nonlinear solution that is known today is that of a stationary zero potential vorticity lens on an f plane (see e.g., Rossby, 1948, Nof, 1981a, b). In this brief paper, we shall derive a new exact solution that complements the known solution mentioned above; in contrast to the stationary lens, however, we shall be concerned with a lens that propagates along a circle. In a sense our analysis simply adds inertial oscillations to the known steady lens solution. Recall, however, that such an addition *does not represent a simple superposition* because the latter can only be done with *linear processes* such as those obeying the Laplace equation.

Our method of solution is as follows. First, we shall transfer the familiar shallow water equations to a coordinate system that travels along a circle. With the aid of this transformation the momentum equations are separated into two parts, a time-dependent part and a stationary part both of which are easily solved.

2. Transformed equations

The governing equations in a coordinate system traveling with the lens at a constant speed C (Fig. 1) will now be derived. One begins by transforming the conventional "reduced gravity" shallow water equations,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g'\frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g'\frac{\partial h}{\partial y} = 0, \tag{2.1a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0,$$

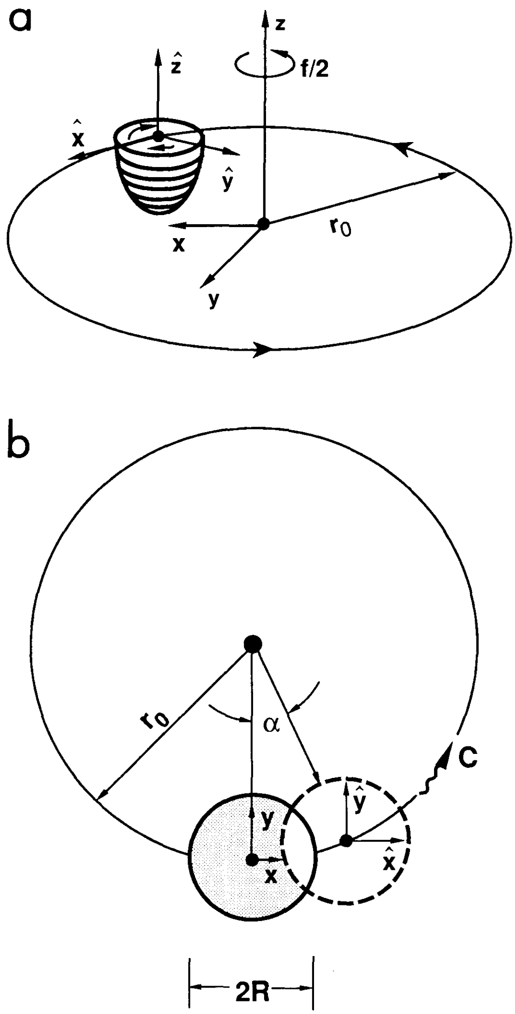

Fig. 1. (a) A 3-dimensional view of the orbiting lens; (b) A schematic top view of the drifting lenses. The migration speed C is positive for cyclonic motion. The dashed circle shows the eddy's position after a brief period of migration; the "wiggly arrow" indicates migration.

to a new $(\hat{x}, \hat{y})$, coordinate system that remains parallel to the old fixed coordinates x, y but moves around the circle r_0 (Fig. 1b). The notation in (2.1a) is conventional: u and v are the horizontal two-dimensional velocity components in the x and y direction, g' the "reduced gravity" ($g\,\Delta\rho/\rho$ where $\Delta\rho$ is the density difference between the layers), f the Coriolis parameter, and h is the total depth. As shown in Nof (1984), the appropriate transformation is,

$$\hat{x} = x - \int_0^t C_x\,\mathrm{d}t, \qquad \hat{y} = y - \int_0^t C_y\,\mathrm{d}t,$$
$$\hat{u} = u - C_x, \qquad \hat{v} = v - C_y, \qquad (2.1\text{b})$$
$$\hat{t} = t, \qquad \hat{h} = h,$$

where $\hat{h}$ is the depth, C_x and C_y are the translation speeds in the $\hat{x}$ and $\hat{y}$ directions (identical to the x and y directions) *which are taken to be functions only of time*, and the "hatted" variables correspond to the new moving system. Since the old time t is now a function of the new variables $\hat{x}$, $\hat{y}$ and $\hat{t}$ i.e.,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \hat{x}}\left(\frac{\partial \hat{x}}{\partial t}\right) + \frac{\partial}{\partial \hat{y}}\left(\frac{\partial \hat{y}}{\partial t}\right) + \frac{\partial}{\partial \hat{t}}\left(\frac{\partial \hat{t}}{\partial t}\right),$$

the time and space derivatives corresponding to (2.1) are,

$$\frac{\partial}{\partial t} = -C_x\frac{\partial}{\partial \hat{x}} - C_y\frac{\partial}{\partial \hat{y}} + \frac{\partial}{\partial \hat{t}},$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \hat{x}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \hat{y}}. \qquad (2.2)$$

In terms of the new variables (defined by (2.1b)) and the corresponding relationships (2.2), the governing "reduced-gravity" shallow water equations (2.1a) are,

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{\partial C_x}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} - f(\hat{v} + C_{\hat{y}}) + g'\frac{\partial \hat{h}}{\partial \hat{x}} = 0,$$
$$\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{\partial C_y}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + f(\hat{u} + C_{\hat{x}}) + g'\frac{\partial \hat{h}}{\partial \hat{y}} = 0, \qquad (2.3)$$
$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{h}\hat{u}) + \frac{\partial}{\partial \hat{y}}(\hat{h}\hat{v}) = 0,$$

where $\hat{u}$ and $\hat{v}$ are the horizontal velocity components in the $\hat{x}$ and $\hat{y}$ directions. Note that, so far, no statements have been made regarding $C_{\hat{x}}$ and $C_{\hat{y}}$ save the fact that they are time-dependent.

We now proceed and state that C is a constant and is directed along a circle with a radius r_0 so that the migration rates along the $\hat{x}$ and $\hat{y}$ axes are,

$$C_{\hat{x}} = C\cos(C\hat{t}/r_0)$$
$$C_{\hat{y}} = C\sin(C\hat{t}/r_0). \tag{2.4}$$

Introducing the above relationship to (2.3), one finds,

$$\frac{\partial \hat{u}}{\partial \hat{t}} - \frac{C^2}{r_0}\sin(Ct/r_0) + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} - f[\hat{v} + C\sin(C\hat{t}/r_0)] + g'\frac{\partial \hat{h}}{\partial \hat{x}} = 0,$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} - \frac{C^2}{r_0}\cos(C\hat{t}/r_0) + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + f[\hat{u} + C\cos(C\hat{t}/r_0)] + g'\frac{\partial \hat{h}}{\partial \hat{y}} = 0, \tag{2.5}$$

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{h}\hat{u}) + \frac{\partial}{\partial \hat{y}}(\hat{h}\hat{v}) = 0,$$

which is our desired set of equations.

3. Migration rate

It is further assumed that only $C_{\hat{x}}$ and $C_{\hat{y}}$ are time dependent; the lens structure is taken to be constant. Under such conditions (2.5) gives a balance between the Coriolis force and the centrifugal acceleration,

$$\frac{C^2}{r_0}\sin(C\hat{t}/r_0) + fC\sin(C\hat{t}/r_0) = 0, \tag{3.1a}$$

$$\frac{C^2}{r_0}\cos(C\hat{t}/r_0) + fC\cos(C\hat{t}/r_0) = 0, \tag{3.1b}$$

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial \hat{v}}{\partial \hat{t}} = \frac{\partial \hat{h}}{\partial \hat{t}} \equiv 0. \tag{3.1c}$$

There are then two solutions for the migration rate,

$$\boxed{C = 0, \qquad C = -fr_0.} \tag{3.2}$$

The first ($C = 0$) is the trivial solution. The second ($-fr_0$) corresponds to an anticyclonic migration and will be the focus of our examination. Substitution of (3.2) back into (2.5) gives the familiar symmetric shallow water equations for a stationary nonlinear feature,

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} - f\hat{v} + g'\frac{\partial \hat{h}}{\partial \hat{x}} = 0,$$

$$\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} + f\hat{u} + g'\frac{\partial \hat{h}}{\partial \hat{y}} = 0 \tag{3.3}$$

$$\frac{\partial}{\partial x}(\hat{h}\hat{u}) + \frac{\partial}{\partial y}(\hat{h}\hat{v}) = 0.$$

These equations possess the following exact solution for a lens with zero potential vorticity ($\partial \hat{v}/\partial \hat{x} - \partial \hat{u}/\partial \hat{y} + f = 0$),

$$\boxed{\begin{aligned} &\hat{u} = \frac{f\hat{y}}{2}, \qquad \hat{v} = -\frac{f\hat{x}}{2}, \\ &\hat{h} = \bar{h} - \frac{f^2}{8g'}(\hat{x}^2 + \hat{y}^2) \end{aligned}} \tag{3.4}$$

where $\bar{h}$ is the lens depth as its center [$\hat{h}(0, 0)$]. We see that the migrating zero potential vorticity lens is identical in structure to the stationary lens. Approximate lens solutions of the kind discussed by Csanady (1979) and Flierl (1979) for finite potential vorticity are also, of course, possible. Note that our general solution is consistent with the relationship derived by Ball (1963) for the movement of the center of gravity of a fluid lying over a paraboloid.

To check our solution [(3.4) with $C = -fr_0$)], one can transform it back to the usual time dependent coordinate system that we begin with. Namely, one takes,

$$x = \hat{x} - \int_0^t fr_0\cos(f\hat{t})\,d\hat{t} = \hat{x} - r_0\sin(f\hat{t}),$$
$$y = \hat{y} + \int_0^t fr_0\sin(f\hat{t})\,d\hat{t} = \hat{y} - r_0\cos(f\hat{t} + r_0), \tag{3.5}$$

and so on, as stated by (2.1b). One ultimately finds,

$$u=\frac{f}{2}[y-r_0-r_0\cos(ft)],$$

$$v=\frac{-f}{2}[x-r_0\sin(ft)],$$

$$h=\bar{h}-\frac{f^2}{8g'}\{[x+r_0\sin(ft)]^2 + [y+r_0\cos(ft)-r_0]^2\}, \quad (3.6)$$

which satisfies the original set of Eq. (2.1a), as should be the case.

It is hoped that this report will encourage future high-frequency observations of lenses in the ocean. Presently, satellite coverage provides information only once or twice a day and this does not permit an identification of inertial motions (such as our new oscillations) in warm-core rings. In addition, the orbits in question may have a radius of, say, 5 km, a scale that cannot be resolved observationally because of the difficulties in identifying the ring center to this accuracy.

4. Acknowledgments

The critical comments of two anonymous reviewers on an earlier version of this paper which contained some misstatements are very much appreciated. This research was supported by ONR grant number N00014-87-J-1209 and NSF grant number OCE-871103.

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