The Drifting Confluence Zone

Ivan Lebedev* and Doron Nof

Department of Oceanography 3048, The Florida State University, Tallahassee, Florida

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ABSTRACT

Alongshore migration of a western boundary current separation is investigated with a nonlinear inviscid reduced gravity model on an f-plane. Separation due to a collision with an opposing current is considered. It is known that for such a case stationary collision and separation is possible only for boundary currents with "balanced" transports, that is, equal near-wall depths. The authors perturb this stationary solution with a small steplike variation of the opposing current transport and focus on the resulting time-dependent flow. Two different analytical methods to compute the migration rate are used. The first method involves integrated balances, and the second involves the path equation for the separated flow.

Using the first approach, it is found analytically that the flow consists of one current intruding into the area occupied by the other. After an initial adjustment period the intrusion becomes steadily propagating. The width of intrusion is much greater than the width of boundary currents, whereas the migration speed is much smaller than the speed of the currents. The speed of the opposing current intrusion into the main western boundary current is given approximately by the formula \( C \approx (q'H)^{1/2}(D_1^2 - D_2^2)^{1/2} AD_{12} \), where \( H \) is the undisturbed depth of the main poleward flowing current, \( D_1 \) is its depth near the wall (non-dimensionalized by \( H \)), and \( D_2 \) is the opposing current near-wall (dimensional) depth. Due to the nonlinearity of the process, the expression for the opposite speed (i.e., the main western boundary current intrusion into the opposing current) has a similar looking but different form.

Using the second approach, the original initial value problem is reduced to a time-dependent path equation for the separated current. It is shown analytically that, as should be the case, in the limit \( t \to \infty \) the path equation solution is identical to the earlier solution for the steady propagating intrusion. The migration process exhibits an hysteresis; that is, the progression of the separated currents differs from its corresponding regression.

Application of the theory to the collision and separation of the Brazil and the Malvinas Currents is discussed. It is suggested that the observed migrations of the separation latitude may be caused by seasonal changes of the Malvinas Current transport.

1. Introduction

The separation of western boundary currents has been the subject of investigation for the last 50 years. However, existing theories of separation predict a separation that is stationary for given properties of the current. In this article we present a theory of western boundary currents separation that is unsteady, that is, it drifts in the north-south direction.

a. Observational background

This work has been motivated by the observations that the Brazil Current (Fig. 1) separates from the continent in different locations at different times. The separation latitude constantly fluctuates and has been observed (Olson et al. 1988) to change as much as 800 km in a season (Fig. 2).

The Brazil Current is a warm saline current that flows poleward along the continental margin of South America (Fig. 1). At some point off the coast of Argentina and Uruguay it encounters the cold and fresh waters of the Malvinas (Falkland) Current, which flows in the opposite direction. Both currents then turn offshore and form a series of large-scale meanders in a region known as the Brazil–Malvinas confluence. Olson et al. (1988) associate the southernmost front of the Brazil Current with the 21°C isotherm and the northernmost edge of the Malvinas Current with the 16°C isotherm. The transition area between these two fronts is filled with mesoscale eddies and extends far into the ocean interior. This region is now recognized as one of the most energetic zones in the World Ocean and recently has become the subject of intense study. An excellent review of observational results in this area is given by Peterson and Stramma (1991). Hydrographic data is reported in Reid et al. (1977), Gordon (1981), Gordon and Greengrove (1986), Roden

* Current affiliation: Flinders Institute for Atmospheric and Marine Sciences, Flinders University of South Australia, Adelaide, South Australia, Australia.

† Additional affiliation: Geophysical Fluid Dynamics Institute, The Florida State University, Tallahassee, Florida.

Corresponding author address: Prof. Doron Nof, Department of Oceanography 3048, The Florida State University, Tallahassee, FL 32306-3048.

E-mail: nof@ocean.fsu.edu

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Reid et al. (1977) locate the confluence zone in the mean between 36° and 39°S.

Temporal variability in the confluence region was studied using satellite-derived sea surface temperature (SST) data (Olson et al. 1988; Podesta et al. 1991; Provost et al. 1992), satellite altimetry data (Provost and Le Traon 1993), and data obtained from a deployed array of inverted echo sounders (Garzoli and Garaffo 1989; Garzoli 1993). All of these studies reveal temporal changes in the separation latitude of the Brazil Current. Olson et al. (1988) have found that the Brazil Current separates from the continental shelf and slope in the mean at 35.8°S with a standard deviation of 1.1° and a total range of 4.8°. The separation is north of the zero wind-stress curl line, located near 47°S to 48°S. Podesta et al. (1991) analyzed 4 years of NOAA Advanced Very High Resolution Radiometer data and found a strong annual cycle, which explains a high portion of the SST variability in the confluence region. Provost et al. (1992) studied the SST dataset compiled by Olson et al. (1988) and observed the dominance of annual and semiannual signals. Using dynamic height records derived from inverted echo sounders data, Garzoli (1993) observed northward penetration of the Malvinas Current during the austral winter. Garzoli and Garaffo (1989) describe the migration of the front of the order of 100 km within a 12-month period.

b. Theoretical background

The earliest models of wind-driven circulation and separation (Munk 1950; Munk and Carrier 1950) were based on linearized equations of motion and represented the ocean as a layer of constant density. In this case the flow pattern in the basin has a form of two counterrotating gyres with intense currents along the western boundary. Both currents separate from the west coast and form a straight eastward drift at the latitude where the wind stress curl vanishes. Separation of western boundary currents is, therefore, explained by the vanishing of meridional Sverdrup transport in the ocean interior.

An examination of the density stratification in models of western boundary currents provides a different explanation of their separation. Poleward increase of the Coriolis parameter can only be compensated by the growing difference between the upper-layer thickness in the ocean interior and the thickness at the western boundary of the current. Charney (1955) and Morgan (1956) have shown that, consequently, a poleward

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Fig. 1. Averaged mass transport of the upper 1000 m in the western Argentina Basin based on the hydrographic data (adapted from Confluence Principal Investigators, Eos, 1990).
flowing inertial boundary current cannot exist past a critical latitude at which point the depth of the upper layer at the western boundary vanishes. They concluded that at this point the current is forced to separate from the coast and veer offshore. Parsons (1969) incorporated this mechanism of upper-layer outcropping in a model of wind-driven circulation in a closed basin. Kamenkovich and Reznik (1972) extended Parsons’ (1969) results to a two-layer model. Moore and Niiler (1974) gave a detailed analysis of physical balances in the vicinity of the separation point in the framework of a purely inertial reduced gravity model.

Another explanation of the separation relates this phenomenon to the topography of the continental slope and margin. It was shown that both changes of the bottom slope (Warren 1963; Greenspan 1963) and irregularities of the coastline geometry (Stern and Whitehead 1990) may cause the boundary current to separate from the coast. It is believed that the separation of the Gulf Stream occurs under strong topographic control (Olson et al. 1988), but this is probably not the case with the South Atlantic.

A different approach to the separation problem was recently proposed by Agra and Nof (1993, hereafter referred to as AN). In their model, the western boundary current is driven offshore by the impact of the countercurrent flowing in the opposite direction. Note that this current also appears in the double-gyre models of the general circulation (Munk 1950; Munk and Carrier 1950; Cessi 1990, 1991) but, due to the absence of nonlinearity, it plays a passive role in the separation process. The separation mechanism of AN is independent of β, wind stress field, or basin geometry. Nof (1993) has shown that in the presence of β the system of colliding currents together with the separated current has a tendency to drift uniformly along the coast. Agra and Nof established that, even on an f plane, the collision cannot be steady. Steady separation can occur only for a limited set of opposing currents with matching properties. Since in most cases this matching is not achieved, they predicted that this should lead to non-steady separation and temporal displacement of the separation point along the western boundary.

For a review of recent numerical studies of western boundary currents separation the reader is referred to Ierley (1990), Cessi et al. (1990), Cessi (1991), Chassignet and Bleck (1992), and the references mentioned therein. Modeling studies of the general circulation in ocean basins with realistic numerical models (i.e., including bottom topography, geography of the coastline, parameterization of air–sea interaction, and driven by observed winds) are numerous and need not all be reviewed here. We shall only mention some of these studies that deal with space–time variability of the separation. Thompson and Schmitz (1989) have found that the position of the Gulf Stream depends on the strength of the opposing deep western boundary current. Numerical experiments indicate that the separation latitude of the Brazil Current depends on mass transports of the Brazil and the Malvinas Currents (Matano 1993) and experiences changes due to wind induced variations in both transports (Matano et al. 1993; Smith et al. 1994).

c. Present approach

We shall study the separation of a western boundary current in the framework of a baroclinic ‘‘reduced gravity’’ model on an f plane and consider separation of a western boundary current due to the impact of the
opposing current. We shall assume that in the local separation dynamics the role of the ocean interior and wind forcing is minor. Implicitly, however, wind forcing is, of course, present in our model as the agent that sets up the interacting currents in the first place. The novelty of our approach is that we shall consider separation as a nonstationary process and look for time-dependent solutions to the shallow-water equations. We shall address a fully nonlinear problem that is not restricted to being quasigeostrophic.

Consider two currents with equal densities flowing above an infinitely deep layer of a denser fluid. The currents flow along a vertical western wall in opposite directions, collide at some point along the wall, and veer offshore in the form of a single separated jet (Fig. 3). For simplicity, we shall first address the problem in the Northern Hemisphere. We shall refer to the poleward flowing current as the main current and to the southward current as the counter (or opposing) current. We shall assume that the potential vorticity of the main current is uniform and that the upper-layer depth reaches a constant value $H$ far away from the western wall. Initially we shall assume that the potential vorticity of the countercurrent is zero and that the corresponding interface strikes the surface of the ocean at some distance from the wall. It will become clear later that the choice of the potential vorticities has no significant influence on the problem.

Agra and Nof's results show that a stationary solution exists only for colliding currents that are "balanced." Balanced currents satisfy a condition that their upstream near-wall depths and velocities are equal, which is necessary for the conservation of mass in the system. We shall perturb a steady solution of AN by introducing a small variation to the countercurrent transport and focusing on the resulting time-dependent flow, which is a slow adjustment on an $f$ plane resulting from a nonlinear advective process. Our aim is to understand the associated dynamics of the nonstationary separation.

First we shall assume (and later confirm) that the flow at $t \to \infty$ has a form that is steady in a moving coordinate system. Next we shall assume that the obtained migration speed and width for $t \to \infty$ are the characteristic temporal and spatial scales of the flow not only at $t \to \infty$ but also during the entire process of nonstationary separation. This will allow us to tie a theory of thin jets (Cushman-Roisin et al. 1993) to our seemingly different problem of collision and separation. As a result, we shall reduce the problem to the solution of the path equation for a separated jet with specially chosen boundary conditions. Numerical integration of the path equation will allow us to obtain the complete time-dependent solution to the problem.

The approach that we shall use here is not the classical matching asymptotic expansions, primarily be-

![Fig. 3. Schematic three-dimensional diagram of a western boundary current separated due to an opposing southward flow. For simplicity, we first address the problem as if it were to occur in the Northern Hemisphere. The opposing jet has a density identical to that of the main western boundary current. Most currents in the ocean are expected to be unbalanced so that the system would drift along the coast at speed $C$ (positive for a migration toward the equator). The "wiggy" arrow denotes migration and should be distinguished from solid arrows that denote actual current speeds. Adapted from Agra and Nof (1993).](image-url)
cause the exact solution in the near-wall region is not known. We shall use a different technique where we enclose the near-wall region, together with the transition region, in the control volume and obtain the matching parameters from the conservation of mass and momentum within the control volume. The stationary flow approximation asymptotically applies within the whole control volume whereas the thin jet theory applies only close to its offshore boundary.

The paper is organized as follows. The formulation of the problem is discussed in section 2. Section 3 addresses the asymptotic flow at \( t \to \infty \). Section 4 includes a discussion of the new spatial and temporal scales and the application of the path equation to the problem. Section 5 contains applications of the theory to the South Atlantic, discussion, and conclusions. Our numerical method of solution of the path equation is discussed in appendix A. A list of symbols is given in appendix B.

2. Formulation

Suppose that the equilibrium of the given balanced currents is altered so that the countercurrent depth and velocity take new fixed values \( H_2 \) (\( \neq H_0 \)) and \( V_2 \) (\( \neq V_0 \)) (where \( H_0 \) and \( V_0 \) are the near-wall depth and speed of the balanced state, and \( H_2 \) and \( V_2 \) are the countercurrent near-wall depth and speed). The response of the system to such a perturbation consists of two processes with different timescales and physical natures.

The immediate response consists of (i) Kelvin waves that propagate upstream along the wall and change the upstream velocity and depth of the main current and (ii) short gravity waves that propagate downstream and alter the downstream separated current (e.g., see Paldor 1983). The existence of these waves is not only intuitively expected. It is clearly seen in the numerical simulations described in Lebedev and Nof (1996, manuscript submitted to Deep-Sea Res.).

This first stage of the temporal evolution can be called a geostrophic adjustment. It will become clear later that, in general, the resulting adjusted near-wall depth \( H_1 \) (\( \neq H_0 \)) of the main current is not necessarily equal to the disturbed near-wall depth \( H_2 \) (\( \neq H_0 \)) of the countercurrent. Therefore, the adjustment cannot lead to stationary separation; instead, a slow evolution of the adjusted flow field occurs afterward. This is the second stage of the temporal evolution. It involves curving of the separated current and migration of the separation point along the wall and is the main focus of our work.

We shall use a Cartesian coordinate system with the \( x \) axis directed seaward normal to the wall and the \( y \) axis directed northward parallel to the wall (Fig. 4). The origin of the coordinate system coincides with the location of the separation point at time \( t = 0 \). Recall that the stationary flow (AN) has the relatively simple form of a separated jet following a straight line at an angle \( \theta_0 \) to the wall (Fig. 4).

\[ a. \text{Detail of the flows in the various regions} \]

1) Section AB (Main Current)

The velocity and depth distributions in a one-dimensional current with constant potential vorticity have the form:

\[
\begin{align*}
\nu_0 &= V_0 e^{-x/R_d}, \quad y \to -\infty \\
\nu_0 &= H_0 - (H - H_0)e^{-x/R_d}, \quad y \to -\infty \\
u_0 &= 0, \quad y \to -\infty,
\end{align*}
\]

where \( u_0(x, y), v_0(x, y) \), and \( h_0(x, y) \) are the dynamical fields at \( t = 0 \); \( H \) is the undisturbed depth at \( x \to \infty \); and \( R_d = (g'H)^{1/2}/f \) is the Rossby radius. The remaining notation is conventional; for clarity, all variables are defined in both the text and appendix B. The near-wall velocity \( V_0 \) and depth \( H_0 \) are linked by the formula,

\[
H_0 = H \left(1 - \frac{V_0}{f R_d}\right),
\]

which follows from the geostrophic relationship.

2) Section CD (Countercurrent)

The velocity and depth distributions for the zero potential vorticity current are given by

\[
\begin{align*}
\nu_0 &= -V_0 - fx, \quad y \to +\infty \\
h_0 &= H_0 - \frac{f}{g'} V_0 x - \frac{f^2}{2g'} x^2, \quad y \to +\infty \\
u_0 &= 0, \quad y \to +\infty,
\end{align*}
\]

which implies that the interface strikes the surface at a distance

\[
\gamma_0 = \frac{1}{f} \left[-V_0 + (V_0^2 + 2g'H_0)^{1/2}\right],
\]

where \( V_0 \) and \( H_0 \) are the velocity and depth at the wall.

3) Section EF (Separated Current)

The velocity and depth distributions of the offshore current are a combination of two boundary currents with matched pressures and velocity at the separating streamline. In the local tilted coordinate system \((x', y')\) associated with the separated current (Fig. 3), they have the form:

\[
\begin{align*}
u_0' &= 0 \\
\nu_0' &= \begin{cases} V_0 - fx' & \text{if } x' < 0 \\ V_0e^{-x'R_d} & \text{if } x' > 0 \end{cases}
\end{align*}
\]
The veering angle $\theta_0$ is
\[ \sin\theta_0 = \frac{M_{10} - M_{20}}{M_{10} + M_{20}}, \]  \hfill (2.12) 

where
\[ M_{10} = \int_0^\infty [h_0 v_0^2]_{y=-\infty} dx \]  \hfill (2.13) 
is the momentum flux of the main current and
\[ M_{20} = \int_0^\infty [h_0 v_0^2]_{y=+\infty} dx \]  \hfill (2.14) 
is the momentum flux of the countercurrent. Recall that the subscript zero denotes association with the balanced state (i.e., $t = 0$). This completes the presentation of the AN solution, which corresponds to our initial conditions for the migration problem.

\textbf{b. Boundary conditions for the time-dependent migration problem}

1) NO NORMAL FLOW AT THE WESTERN WALL 
\[ u = 0 \quad \text{at} \quad x = 0. \]  \hfill (2.15) 

2) PRESCRIBED INFLOW IN THE UPSTREAM COUNTERCURRENT ($y \to +\infty$) 
At time $t = 0$ the transport and velocity of the countercurrent are changed and maintained at a new level for all $t > 0$. The prescribed velocity and depth distributions have the same general form as (2.5)–(2.8). For clarity, the new depth and velocity at the wall will be denoted by $H_2$ and $V_2$. We shall take the change of the upstream velocity and depth to be small relative to the intrinsic scales of both variables $[(H_2 - H_0)/H \ll 1$ and $(V_2 - V_0)/(fR_0) \ll 1]$, and define
\[ \epsilon = (H_2 - H_0)/H \ll 1. \]  \hfill (2.16) 

3) OPEN BOUNDARY CONDITIONS IN THE UPSTREAM MAIN CURRENT ($y \to -\infty$) 
The Kelvin wave, generated by the perturbation of the countercurrent and propagating upstream, changes
the initial velocity and depth distributions (2.1)–
(2.4). Since the Kelvin wave does not alter the poten-
tial vorticity of the current, the velocity and depth dis-
tributions still have the form:

\[ u = 0, \quad y \to -\infty \]  \hspace{1cm} (2.17)

\[ v = V_1 e^{-x/R_d}, \quad y \to -\infty \]  \hspace{1cm} (2.18)

\[ h = H - (H - H_1) e^{-x/R_d}, \quad y \to -\infty, \]  \hspace{1cm} (2.19)

where \( V_1 \) and \( H_1 \) are the unknown velocity and depth at the wall; they are related by

\[ H_1 = H \left(1 - \frac{V_1}{R_d}\right). \]  \hspace{1cm} (2.20)

Computation of \( H_1 \) and \( V_1 \) is a part of the solution to the initial value problem.

4) Open boundary conditions in the downstream separated current \( (x \to \infty) \)

The distributions of depth and velocity in this region still have the general form (2.9)–(2.11) with the new velocity \( (V_3) \) and depth \( (H_3) \) in place of the balanced velocity \( (V_0) \) and depth \( (H_0) \) along the potential vorticity front. The computation of \( V_3 \) and \( H_3 \) is a part of the solution to the problem.

c. Mass budget of the stationary flow

To demonstrate that the geostrophically adjusted flow cannot be stationary, we shall, temporarily, as-
sume that the flow is stationary and consider its mass balance. The main current is geostrophic and has a mass transport

\[ T_1 = \frac{g'}{2f} (H^2 - H_1^2), \]

which is always positive. Similarly, the mass transport of the countercurrent is

\[ T_2 = \frac{g'}{2f} H_2^2, \]

and a straight separated current has a mass transport of

\[ T_3 = \frac{g'}{2f} H^2. \]

In the stationary state, \( (T_1 + T_2) \) must be equal to the transport of the separated current; that is,

\[ H^2 - H_1^2 + H_2^2 = H^2, \]  \hspace{1cm} (2.21)

implying that \( H_1 = H_2 \), which is the AN criteria of balanced currents. The above formulas state that, while the transpors of boundary currents can vary, the geo-

strophic transport of the separated current is fixed (and depends only on the constant undisturbed upper-layer depth \( H \)). Therefore, when the total transport of bound-

ary currents differs from the value \( g'H^2/2f \), a nonsta-
tionary flow must take place.

The nonstationary flow can be computed as a solution to the shallow-water equations, together with the initial conditions \( u|_{t=0} = u_0(x, y), \ v|_{t=0} = v_0(x, y) \), and the boundary conditions (2.15)–(2.17), (2.18)–
(2.20). We now proceed to solve this problem with a perturbation technique.

3. Asymptotic solution at \( t \to \infty \)

Consider the final stage of a nonstationary collision and separation, that is, the solution of the initial value problem at \( t \to \infty \). We shall assume that at \( t \to \infty \) the flow has a form of “steady intrusion.” We shall use the balance of integrated mass and momentum and a perturbation scheme in \( \epsilon \), the ratio between the depth increase and the balanced depth (2.16), to compute its speed and width. We shall establish a new length scale \( L \sim R_0/\epsilon \), fundamental to the nonstationary separation. The order of magnitude difference between the new “collision” length scale \( L \) and the intrinsic “reduced gravity” length scale \( R_0 \) will allow us later to derive a solution uniformly valid in time.

a. Governing equations

Suppose that, as a result of the perturbation, the up-
stream transport and velocity of the countercurrent have increased. It is expected that the flow field (at \( 0 < t < \infty \)) will have a form of the stronger countercurrent intrusion into the weaker main current.

In the moving coordinate system, the steadily propagating flow (Fig. 5) is described by the modified “reduced gravity” equations:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + f v = -g' \frac{\partial h}{\partial x} - fC \]  \hspace{1cm} (3.1)

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g' \frac{\partial h}{\partial y} \]  \hspace{1cm} (3.2)

\[ \frac{\partial}{\partial x} (uh) + \frac{\partial}{\partial y} (vh) = 0, \]  \hspace{1cm} (3.3)

where the term \(-fC\) on the right-hand side of (3.1) is the Coriolis force acting on a motionless fluid parcel due to its translation with the coordinate system. The Bernoulli integral has the form

\[ B(\psi) = \frac{1}{2} (u^2 + v^2) + g' h + fC x, \]  \hspace{1cm} (3.4)

where the streamfunction is defined in the normal manner \((\psi = \partial \psi/\partial x, h u = -\partial \psi/\partial y)\).

As mentioned, our aim is to determine the migration speed \( C \) and the intrusion width \( L \). This will be achieved by considering the mass and momentum balance within the control volume ABFEDCA (Fig. 5), which is chosen large enough so that the currents en-
Here circles indicate counterclockwise contour integration.

Since \( u = 0 \) on both the wall and BF, the first term in (3.5) drops out and the integrated momentum equation takes the form:

\[
\oint (-hv^2 + f \psi - \frac{g'}{2} h^2) dx = 0. \tag{3.6}
\]

This equation can be further simplified by calculating the expression for the streamfunction \( \psi \). For a geostrophic flow across CD, EF, and AB, (3.1) reduces to

\[
v = \frac{g'}{f} \frac{\partial h}{\partial x} + C. \tag{3.7}
\]

Multiplication of (3.7) by \( h \) and integration in \( x \) gives

\[
\psi = \int_0^x h dx = \frac{g'}{2f} h^2(x, y)
\]

\[
- \frac{g'}{2f} h^2(0, y) + C \int_0^x h dx, \tag{3.8}
\]

where we have set \( \psi = 0 \) along the boundary \( x = 0 \). In this formula, \( h(0, y) \) is the near-wall upper-layer depth at distance \( y \) from the stagnation point 0.

We now substitute (3.8) into (3.6) to find

\[
\oint \left[-hv^2 + fC \int_0^x h dx - \frac{g'}{2} h^2(0, y)\right] dx = 0,
\]

which includes velocity and depth distributions across CD, EF, and AB where the flow has a simple one-dimensional structure.

c. Integrated mass balance

Integrating the continuity equation (3.3) over the control volume ABFEDCA, using the Stokes theorem and noting that \( u = 0 \) on eastern and western sides of the control volume, we obtain the integrated mass balance in the form:

\[
\oint h u dx = 0. \tag{3.10}
\]

By using the geostrophic relation, (3.10) can be transformed to

\[
\frac{g'}{2f} (H_2^2 - H_1^2) + C \oint h dx = 0, \tag{3.11}
\]

where \( H_1 \) and \( H_2 \) are the upstream depths of the main current and the countercurrent, respectively (Fig. 5).
d. Upstream and downstream structure of the flow

The velocity and depth distributions across AB and CD are specified by the boundary conditions discussed in section 2b. From potential vorticity conservation, it follows that the velocity and depth distributions in the downstream separated current (section EF) have the form

\[
v = \begin{cases} 
V_3 + C - f(x - L) & \text{when } L - \gamma_3 < x < L \\
V_3 e^{-(x-L)/R_d} + C & \text{when } x > L 
\end{cases}
\]  
(3.12)

where \( \gamma_3 = f^{-1}[-V_3 + (V_3^2 + 2g\gamma' H_3)^{1/2}] \) is the width of the zero potential vorticity part of the separated current, \( L \) the width of the intrusion, and \((V_3 + C) \) and \( H_3 \) are the velocity and depth along the downstream separating streamline. As before, \( V_3 \) and \( H_3 \) are connected by

\[
H_3 = H \left( 1 - \frac{V_3}{f R_d} \right). 
\]  
(3.14)

Since the separating streamline can be traced to the wall, the unknown values of \( V_3 \) and \( H_3 \) can be related to the known values of \( V_2 \) and \( H_2 \) by the Bernoulli integral,

\[
\frac{1}{2}(V_2 - C)^2 + g'H_2 = \frac{1}{2}(V_3 + C)^2 + g'H_3 + fCL.
\]  
(3.15)

Similarly, the unknown values \( V_1 \) and \( H_1 \) of the main current near-wall velocity and depth can be linked to the known \( V_2 \) and \( H_2 \),

\[
\frac{1}{2}(V_2 - C)^2 + g'H_2 = \frac{1}{2}(V_1 + C)^2 + g'H_1.
\]  
(3.16)

Solution of the six simultaneous algebraic equations (2.20), (3.9), (3.11), (3.14) – (3.16) with the six unknowns \( C, L, H_1, V_1, H_3, \) and \( V_3 \) allows one to compute the desired speed and width of the intrusion.

e. Solution with a perturbation technique

We shall first look at the order of magnitude of the various terms in the mass and momentum balance equations and deduce the appropriate scales. Before presenting this scale analysis it is recalled that (i) within the jets \( u \sim O(fR_d) \) and that (ii) the width of the jets is \( O(R_d) \). We now assume that, when the perturbation of the countercurrent is small [i.e., \((H_2 - H_0)/H \equiv \epsilon \ll 1, (V_2 - V_0)/(f R_d) \sim O(\epsilon) \ll 1 \)], the migration speed \( C \) is also small and the width of the intrusion is much larger than the width of the jets (i.e., \( C \ll f R_d \) and \( L \gg R_d \)).

Consider now the integrated momentum equation (3.9) subject to the above assumptions. Since the separated jet now flows parallel to the wall (instead of leaving it at an angle \( \theta_0 \), which is the case in the initial steady state), the zeroth-order terms in the integral do not cancel each other. The integral is approximately equal to the sum of the momentum flux carried in and out of the control volume by the countercurrent,

\[
\iint h v^2 \, dx = \int_{\gamma_3}^\gamma [h v^2]_{y=+\infty} dx 
+ \int_{L-\gamma_3}^{L} [h v^2]_{y=-\infty} dx \sim O(Hf^2 R_d^3).
\]

When \( L \gg R_d \), the leading order contribution from the remaining two terms in (3.9) is \( O(fCHL^2) \). It follows that to balance the equation, it is necessary that \( CL^2 \sim fR_d^3 \). Similarly, from the mass conservation equation it follows that \( CL \sim (g'/f)(H_1 - H_2) \sim \epsilon fR_d^3 \). From the above formulas, we finally obtain the appropriate scaling for the two unknowns \( C \) and \( L \),

\[
C \sim O(\epsilon^2 f R_d), \quad L \sim O(R_d/\epsilon).
\]  
(3.17)

On this basis, we nondimensionalize the lengths with \( R_d \), the speed with \( f R_d \), and the depths with \( H \). Next, we expand all the nondimensional variables in a power series in \( \epsilon \); for example,

\[
\begin{align*}
V_1^* &= V_0^* + \epsilon V_1^{(1)} + O(\epsilon^2) \\
H_1^* &= H_0^* + \epsilon H_1^{(1)} + O(\epsilon^2) \\
L^* &= \epsilon^{-1} L^{(-1)} + O(\epsilon^{-1}) \\
C^* &= \epsilon^2 C^{(2)} + O(\epsilon^2) \\
V_2^* &= V_0^* + \epsilon V_2^{(1)} + \cdots
\end{align*}
\]  
(3.18)

where

\[
V_2^{(1)} = \frac{V_2 - V_0}{\epsilon f R_d}
\]

is a known value of order unity.

Substitution of (3.18) into (2.20) and (3.14) yields the zeroth-order balance,

\[
H_1^{(1)} = -V_1^{(1)}; \quad H_0^* = 1 - V_0^*; \quad H_3^{(1)} = -V_2^{(1)}.
\]  
(3.19)

Similarly, substituting (3.19) into (3.16) we obtain that the zeroth-order balance is automatically satisfied, while the first-order balance reads
\[ H_0^* H_1^{(1)} = 1 + V_0^* V_2^{(1)}. \]  
(3.20)

Also, substitution of (3.18) into (3.15) yields the first-order balance,
\[ H_0^* H_3^{(1)} + C(2) L^{(-1)} = 1 + V_0^* V_2^{(1)}. \]  
(3.21)

The next step is to substitute (3.18) into the distributions of velocity and depth across the sides of the control volume [i.e., (2.16), (2.17), (2.18), (2.19), (3.12), and (3.15)], then substitute these distributions into the integrated balances of mass and momentum. Using the "Mathematica" software package (Wolfram 1998) to do the algebra, we find that the leading-order balances are
\[ H_0^* (1 - H_1^{(1)}) + C(2) L^{(-1)} = 0 \]
and  
\[ \frac{1}{2} C(2) L^{(-1)^2} = 2M_{20}^*, \]  
(3.22)

where
\[ M_{20}^* = -\frac{1}{10} (\gamma_0^*)^4 - \frac{1}{2} (\gamma_0^*)^4 V_0^* \]
\[ + \left( \frac{1}{3} H_0^* (V_0^*)^2 \right) (\gamma_0^*)^3 \]
\[ + \left( H_0^* (V_2^*)^2 - \frac{1}{2} (\gamma_0^*)^3 \right) (\gamma_0^*)^2 \]
\[ + \int_0^{\gamma_0^*} \left[ H_0^* (V_2^*)^2 \right]_{\gamma_0^*}^{\gamma_0^*} d\gamma. \]

is the nondimensional momentum flux of the countercurrent in the initial steady state.

From (3.20)–(3.22) all the unknowns can now be easily determined:
\[ H_1^{(1)} = (1 + V_2^{(1)} V_0^*)/H_0^*; \quad V_1^{(1)} = -H_1^{(1)}; \]
\[ H_3^{(1)} = 1, \quad V_3^{(1)} = -1 \]  
(3.23)

\[ L^{(-1)} = 4M_{20}^* V_0^* (V_2^{(1)} + 1)^2 \]
\[ C(2) = \frac{4M_{20}^*}{V_0^* (V_2^{(1)} + 1)^2}. \]  
(3.24)

Note that all the computed coefficients of the asymptotic expansions are of \( O(1) \) as should be the case.

Transformation to the original dimensional variables yields the desired relationships,
\[ H_1 = H_0 + \frac{H}{H_0} \left[ H_2 - H_0 + \frac{V_0 (V_2 - V_0)}{g'} \right]; \]
\[ H_3 = H_2 \]  
(3.25)

\[ C = \frac{(g')^2}{16 f H M_{20}^*} (H_1^2 - H_2^2)^2; \]
\[ L = \frac{8M_{20}^*}{g'} (H_1^2 - H_2^2)^{-1}. \]  
(3.26)

where \( M_{20}^* \) is the momentum flux of the countercurrent defined by (2.14) (i.e., \( \int_0 H_0 v^* y dx \)). For simplicity, we chose to present (3.26) in a way that actually contains terms of higher order, which can be neglected. The corresponding values of \( M_2 \) for zero and finite potential vorticity \((f/H_2, \text{where} \ H_2 \text{is the near-wall depth})\) countercurrents are shown in Fig. 6. Using the dashline approximation, (3.26) can be written approximately as

\[ C \approx \frac{(g')^3}{16 f H M_{20}^* (H_1^2 - H_2^2)^2}; \]
\[ L \approx 2R_0 (H_1^2)^{3/2} (H_1^2 - H_2^2). \]
\[ H_2^* > H_1^*. \]  
(3.27)

It is recalled that the above formula describes the intrusion of the countercurrent into the main current (which follows the increase of the countercurrent's transport with respect to its balanced value). The reduction of the countercurrent's transport at \( t = 0 \) leads to the opposite process, that is, the intrusion of the main current into the countercurrent. It can be shown that the asymptotic migration speed of the steadily propagating intrusion in this case is given by

\[ C = -\frac{(g')^2}{16 f H M_{20}^*} (H_2^2 - H_1^2)^2; \quad H_2 > H_1, \]  
(3.29)

which, in contrast to (3.26), contains the momentum flux \((M_{20}^*)\) of the main current in the denominator. For the main current with the uniform potential vorticity, (3.29) can be written explicitly as

\[ C = -\frac{(g')^2}{16 f H M_{20}^*} \left( H_2^2 - H_1^2 \right)^2; \quad H_2 > H_1. \]  
(3.30)

![Fig. 6. The momentum flux (M20) as a function of the countercurrent near-wall depth (H2) and the current potential vorticity (solid lines). The dashed line is an approximation used for the final representation of the migration speed and intrusion width.](image-url)
Note that it is the nonlinearity of the problem that causes the progression of the opposing current (3.26) to be different from its regression counterpart (3.29).

The above results can be easily generalized to the case where the countercurrent offshore depth is not zero. In this case the migration speed of the intrusion into the main current is

$$C = (g'2/16fM_2/16fM_2)(H_m - H_c),$$  (3.31)

where, as before, $H_1$ and $H_2$ are the near-wall depths of the main and countercurrent; $H_m$ and $H_c$ are the undisturbed offshore depths of the main current and countercurrent, respectively.

f. Discussion

The physical meaning of our solution can be understood if we consider the flow in the fixed reference frame. From this perspective, the potential vorticity front separating the countercurrent from the main current resembles a wide piston slowly moving along the wall. Every instant, the piston displaces ahead of itself the volume of water equal to HCL; this displaced water is deflected sideways and eventually flows along the outer side of the piston. As a result, the mass transport of the main current at the outer side of the potential vorticity front is greater than the transport of the main current ahead of it. The difference of the geostrophic transports given by $(g'2/16f)(H_1 - H_2)$ is equal to HCL, as follows from (3.27)–(3.28).

Let us now consider the expression that relates the near-wall depth $(H_1)$ of the main current to the parameters $(V_2$ and $H_2$) of the countercurrent (3.25). It is recalled that (3.25) was obtained from the Bernoulli equation (3.16) and, therefore, expresses (in a somewhat camouflaged form) the conservation of the Bernoulli energy along the wall. Although, strictly speaking, the principle is valid only in a moving coordinate system, the migration speed $C$ can be neglected as a small variable of a high order, so that the same principle is valid in a fixed coordinate system as well. We suppose that the transition from the original near-wall depth $(H_0)$ of the main current to the new value $(H_1)$ occurs by means of a Kelvin wave during the geostrophic adjustment.

Both physical intuition and inspection indicate that the increase of the transport of the countercurrent should cause an intrusion into the main current. The obtained solution shows that this is a two-stage process: first the Kelvin wave reduces the transport of the main current, and only then the intrusion in the right direction occurs.

Before proceeding to the next section, we can relax the constraints imposed by the particular choice of the potential vorticities of the colliding currents. One can see that in the obtained solution only the value of the adjusted near-wall depth $(H_1)$ of the main current depends on the fact that the main current has a constant potential vorticity. If the value of $H_1$ is considered a given parameter (known from measurements or a numerical model), then the migration speed $C$ depends only on the bulk properties of the flow such as the countercurrent momentum flux. In fact, starting the analysis with the specific velocity and depth distributions of the colliding currents was necessary only to prove that the scale $C \sim O(\epsilon^2fR_d)$ is correct. Once the scale for the migration speed is taken for granted, an identical solution can be derived for the opposing currents with arbitrary distributions of potential vorticity.

4. Solution for $0 < t < \infty$ using the path equation

In this section, we shall introduce a theory of a non-stationary collision valid at all times $0 < t < \infty$. Our goal is to determine the position of the separation point and the shape of the separated current as a function of time. We shall achieve this by reducing the original initial value problem to a much simpler problem for the path equation of a separated current.

a. Scales

We shall see that the scales of the flow allow us to apply the theory of narrow jets (Cushman-Roisin et al. 1993, hereafter referred to as CPR). Consider the flow at an arbitrary time $t < \infty$ (Fig. 7): The asymptotic solution at $t \to \infty$ indicates that it is characterized by two substantially different length scales. The region of strong nonlinear interaction (where the two opposing jets actually collide and push each other seaward) is comparable in size to the width of the jets and, therefore, is of the order of the Rossby radius. Outside this region the flow has a structure of a slowly varying quasi-uniform meandering jet with length scale $R_d/\epsilon$.

Since $R_d/L \sim O(\epsilon) \ll 1$, it follows that at a distance $x \approx O(R_d)$ from the wall the jet falls within the validity regime of CPR’s general theory of thin equivalent barotropic jets. The theory is applicable for arbitrary jets in an unbounded ocean (including those with surfacing interface) provided that the cross-jet length scale is much smaller than the along-jet length scale.

Referring the flow at $t \to \infty$ to a fixed reference frame, we can determine the timescale ($\tau$) for the rate of change of an arbitrary Eulerian variable from the relation $\partial/\partial t = \tau \partial/\partial \tau$. Substituting $R_d$ as an appropriate measure of the distance, we obtain

$$\tau \sim O(R_d/C) \sim O(\epsilon^{-2}f^{-1}).$$

It is logical to consider $\tau$ to be a measure of the entire temporal variability of any Eulerian variable during the slow evolution of the flow.

b. Path equation

The temporal evolution of the separated current outside the near-wall interaction region can be described
by the path equation (CPR) for the potential vorticity front:

\[ C_n = A \frac{\partial^2 \theta}{\partial s^2}, \quad (4.1) \]

where \( C_n \) is the normal velocity of the segment of the front, \( s \) is the distance measured along the front, and \( \theta \) is the angle (measured counterclockwise) between the zonal direction and the tangent to the front. The coefficient \( A \) is defined by

\[ A = \frac{g^2}{f^2 H} \int_{-\infty}^{\infty} h (dh/dn)^2 dn, \quad (4.2) \]

where \( h(n) \) is the depth distribution across the straight portion of the jet. For our separated jet this corresponds to the cross section at \( x \to \infty \), and to leading order, \( A \) can be expressed as

\[ A = \frac{1}{f_0} \frac{M_{10} + M_{20}}{H}, \quad (4.3) \]

where \( M_{10}, M_{20} \) are the momentum fluxes of the colliding currents in the initial stationary state [defined by (2.13) and (2.14)].

As mentioned, the path equation (4.1) is valid only in regions greater than \( R_d \) away from the wall. To apply (4.1) to the problem of collision and separation we need to somehow connect the flow in the near-wall region (where opposing currents flow toward each other and collide) to the flow in the offshore separated current. We shall do that by deriving the boundary conditions [at \( x \sim O(R_d) \)] that will communicate the information about the colliding currents to the path equation. Specification of such boundary conditions is the essence of this section. Once the boundary conditions are derived, they can then be applied at the wall (\( x = 0 \)) since the distance \( \sim O(R_d) \) is much smaller than the length scale \( [O(R_d/\epsilon)] \) implied in the path equation.

The boundary conditions that we shall be looking for are the veering angle \( (\theta) \) and the curvature \((\partial \theta/\partial s)\) of the potential vorticity front at a distance of \( \sim O(R_d) \) away from the wall. The boundary conditions at \( x \to \infty \) can be specified right away since at a large distance from the wall the jet remains undisturbed; that is,

\[ \theta \bigg|_{x=\infty} = \theta \bigg|_{x=0} = \theta_0. \quad (4.4) \]

\[ \frac{\partial \theta}{\partial s} \bigg|_{x=\infty} = 0. \quad (4.5) \]

c. Boundary conditions at the wall

To derive these boundary conditions we shall consider the mass and momentum balance of a "weakly unsteady" flow (in the moving system) in the near-wall region. Consider a "snapshot" of the flow at an arbitrary moment of time (Fig. 8). We choose a control volume large enough so that the inflowing boundary currents are one-dimensional and the outflowing separated current acquires the structure of a narrow jet before it leaves the domain.

We shall consider the weakly unsteady flow in the fixed coordinate system \((x, y)\) defined in section 2. It is recalled that the time dependence of any Eulerian variable is characterized by the timescale \( \tau \sim O(\epsilon^{-2} f^{-1}) \). Therefore, the nondimensional shallow-water equations (2.1)–(2.3) have the usual form with all terms being of \( O(1) \) except the time-dependent terms that are of \( O(\epsilon^2) \). We first take the limit \( x \to 0 \) (no \( u^* \)) to find

\[ \epsilon^2 \frac{\partial u^*}{\partial x^*} + \frac{\partial}{\partial y^*} (v^* u^*/2 + h^*) = 0, \quad (4.6) \]

which can be integrated along the wall [over the nondimensional distance of \( O(1) \)] from the upstream main current (point C) to the upstream countercurrent (point A, Fig. 8) to give

\[ \frac{V_{1*}^2}{2} + H_1^* = \frac{V_{2*}^2}{2} + H_2^* + O(\epsilon^2), \quad (4.7) \]
where \( V_{12}^* \) and \( H_{12}^* \) are the near-wall velocities and depths of the colliding currents. This means that, although the flow in general is time dependent, the near-wall velocities and depths of the boundary currents are still related by the Bernoulli integral because the timescale is long. As mentioned, numerical simulations illustrate that the equilibration of the Bernoulli function along the wall is established during the geostrophic adjustment (Lebedev and Nof 1996, manuscript submitted to Deep-Sea Res.). There is no mechanism by which information can be transmitted to the upstream region of the countercurrent because Kelvin waves can only propagate along a wall on their right-hand side. Therefore, once changed (at \( t = 0 \)), the near-wall Bernoulli function of the countercurrent remains constant. Equation (4.7), which is valid at an arbitrary moment of time, shows that the near-wall Bernoulli function of the main current is also constant. Consequently, the near-wall depth \( (H_1) \) of the main current is time independent.

Integrating the continuity equation over the control volume and using the Stokes theorem, one obtains

\[
\epsilon^2 \int \int \frac{\partial h^*}{\partial t^*} \, dx^* \, dy^* + \oint u^* h^* \, dy^* = 0 \tag{4.8}
\]

which means that up to \( O(\epsilon^2) \) the sum of the mass transports of the colliding currents is equal to the transport of the separated current.

We have seen earlier that the mass transport of a purely geostrophic separated current cannot balance the perturbation of colliding current transports. A physical mechanism that allows mass conservation is the curving of the separated current, which induces an ageostrophic component to its transport. To demonstrate this, we shall follow CPR and introduce a curvilinear orthogonal coordinate system \( (s, n) \) tied to the potential vorticity front (Fig. 8). Note that the veering angle of the separated current varies over a distance of \( O(R_0/\epsilon) \) and, hence, the curvature \( (\partial \theta/\partial s) \) of the potential vorticity front is \( O(\epsilon/R_0) \). We expand the nondimensional along-jet velocity \( (u_s^*) \) and depth \( (h_s^*) \) in the cross section \( EF \) (Fig. 8) in a power series in \( \epsilon \):

\[
u_3^* = u_3^{(0)} + \epsilon u_3^{(1)} + \cdots;
\]

\[
h_s^* = h_s^{(0)} + \epsilon h_s^{(1)} + \cdots, \tag{4.9}
\]

and write the zeroth- and first-order balances of the momentum in the cross-jet direction,

\[
u_3^{(0)} = -\frac{\partial h_3^{(0)}}{\partial n^*}, \tag{4.10}
\]

\[
u_3^{(1)} + K_0(u_3^{(0)})^2 = -\frac{\partial h_3^{(1)}}{\partial n^*}, \tag{4.11}
\]

[see CPR’s (2.20) and (2.25)], where \( K_0 = (R_0/\epsilon) (\partial \theta/\partial s) \) is the nondimensional curvature of the jet. Note that the zeroth-order balance is purely geostrophic and corresponds to the initial stationary state.

Our aim is to compute the mass transport of the jet in cross section \( FE \) (Fig. 8), which is

\[
\int_F h_s^* u_s^* \, dn^* = \int_F h_s^{(0)} u_3^{(0)} \, dn^* + \epsilon \int_F (h_s^{(1)} u_3^{(0)}) d\theta \tag{4.12}
\]

By means of (4.11), the integrand in the second term on the right-hand side can be expressed as

\[
u_3^{(1)} h_3^{(0)} + u_3^{(0)} h_3^{(1)} = -\frac{\partial h_3^{(1)}}{\partial n^*} h_3^{(0)} \]

\[+ u_3^{(0)} h_3^{(1)} - K_0(u_3^{(0)})^2 h_3^{(0)}, \tag{4.13}
\]

and the use of the geostrophic zeroth-order balance yields
\[ u_3^{(1)} h_3^{(0)} + u_3^{(0)} h_3^{(1)} = -\frac{\partial}{\partial r^*} (h_3^{(1)} h_3^{(0)}) - K_0 (u_3^{(0)})^2 h_3^{(0)}. \quad (4.14) \]

We now substitute (4.14) into (4.12) and take the point \( F \) sufficiently far from the potential vorticity front so that \( h_3^*(F) \to 1 \) and \( h_3^{(1)}(F) \to 0 \). The integration then yields

\[ \int_F^E h_3^* u_3^* \, dn^* = \frac{1}{2} - \epsilon K_0 (M_{10}^* + M_{20}^*) + O(\epsilon^2), \quad (4.15) \]

where \( M_{10}^* \) and \( M_{20}^* \) are the nondimensional momentum fluxes of the colliding currents in the stationary state. A return to the dimensional variables gives

\[ \int_F^E h_{3s} u_{3s} \, ds = \frac{g'}{2f} H^2 - M_{10} + M_{20} \frac{\partial \theta}{\partial s}. \quad (4.16) \]

Equating now (4.16) to the sum of the mass transports of the colliding currents,

\[ T_1 + T_2 = \frac{g'}{2f} (H^2 - H_1^2 + H_2^2), \]

one obtains

\[ \frac{\partial \theta}{\partial s} = \frac{g'}{2(M_{10} + M_{20})} (H_1^2 - H_2^2), \quad (4.17) \]

which is the first desired boundary condition for the path equation, which can be applied at the wall \( s = 0 \).

The second boundary condition is the veering angle of the separated current, which is determined by the momentum balance of the flow. Before proceeding to its computation, we define an instantaneous streamfunction of a nonstationary flow

\[ \psi^*(x^*, y^*, t^*) = \int_0^{x^*} h^* u^* \, dx^*. \quad (4.18) \]

From the continuity equation it follows that

\[ h^* u^* = -\frac{\partial \psi^*}{\partial y^*} - \epsilon \int_0^{x^*} \frac{\partial h^*}{\partial t^*} \, dx^*. \quad (4.19) \]

The accuracy with which the mass flux \( (h^* u^*) \) can be derived from the streamfunction alone [i.e., the first term in (4.20)] depends on the distance of the integration in (4.20). For the chosen control volume with the size of \( O(1) \) [i.e., \( O(R_e) \) in nondimensional variables], the definition (4.18) is accurate to within \( O(\epsilon^2) \).

A derivation similar to the one described in section 3 yields the integrated momentum equation in the form,

\[ \epsilon^2 \frac{\partial}{\partial t^*} \int_0^E h^* u^* \, dx^* \, dy^* + \int_0^E h^* u^* v^* \, dy^* \]

\[ - \int_0^E h^* u^*^2 \, dx^* + \int_0^E \psi^* \, dx^* - \epsilon \int_0^E \frac{h^* u^*}{2} \, dx^* \]

\[ - \epsilon^2 \int_0^E \left[ \int_0^{t^*} \frac{\partial h^*}{\partial t^*} \, dx^* \right] \, dx^* \, dy^* = 0. \quad (4.21) \]

This equation reveals an apparent difference between the dynamics of a weakly unsteady flow at different length scales. On the large collision length scale of \( O(1/\epsilon) \), the second time-dependent term yields a leading zeroth-order contribution to the balance of the integrated momentum. However, on the \( O(1) \) scale of the flow (i.e., the flow in the near-wall interaction region), the contribution of both of these terms is only of \( O(\epsilon^2) \) and can be neglected.

It can be seen that the momentum balance in the chosen control volume is dominated by the zeroth-order terms in the distributions of the streamfunction, velocity, and depth across the sides of the control volume. The zeroth-order terms, however, correspond to the initial stationary flow, and the computation of the veering angle of the separated current reduces to the problem considered by AN. Therefore, the veering angle is given by (2.12), and the second boundary condition for the path equation reads

\[ \sin \theta \big|_{s=0} = \sin \theta_0 = \frac{M_{10} - M_{20}}{M_{10} + M_{20}}. \quad (4.22) \]

The initial condition for the path equation is that of a straight jet at \( t = 0 \); that is,

\[ \theta(s) \big|_{s=0} = \theta_0. \quad (4.23) \]

The solution to the path equation (4.1) together with the boundary conditions (4.4), (4.5), (4.17), and (4.22) and the initial condition (4.23) allow one to determine the position of the separation point and the shape of the separated current at any time.

d. Numerical computations

For our numerical solution example we have taken \( f = 10^{-4} \), \( s^{-1} \) and the equilibrium near-wall depth \( H_0 = 0.5H \). The momentum fluxes \( M_{10} = 0.08 f^2 R_e^2 H \) and \( M_{20} = 0.09 f^2 R_0^2 H \) were calculated from the distributions (2.1), (2.2), and (2.5)–(2.8). For the description of the numerical method of the path equation solution the reader is referred to appendix A.

The first example (Fig. 9) shows an evolution of an initially straight separated jet when the near-wall depth difference \( (H_1 - H_2) \) is created at \( t = 0 \) and then kept constant. Note the formation of an intrusion that even-
usually becomes steadily propagating. A plot of the separation point migration speed as a function of time is shown in Fig. 10. The large values of the migration speed at \( t \to 0 \) are clearly due to the discontinuity at \( x = 0 \). Aside from this nonphysical value, the migration speed is seen to be of the same order as its asymptotic value at \( t \to \infty \). The migration speed is seen to reach its asymptotic value after about 90 days, long before the flow can be named a steadily propagating intrusion from the geometrical point of view.

In the other example, we first set \((H_1 - H_2)/H = 0.1\) for the first 180 days of time integration and then \((H_1 - H_2)/H = -0.1\) for the following 180 days (Fig. 11). It is seen that the unbalanced migration is irreversible and the retreat of the separated current during the second half of time integration [curves (d) and (e), Fig. 11] is different from its advance during the first 180 days [curves (b) and (c), Fig. 11]. The location of the separation point after the forcing cycle is completed is not the same as its original position. This qualitatively resembles the phenomenon of hysteresis characteristic for nonlinear systems: after the relaxation of the load, the system does not return to its original configuration.

e. Agreement with the asymptotic solution at \( t \to \infty \)

We shall now demonstrate that the path equation model agrees with the asymptotic solution (at \( t \to \infty \)) presented earlier. To show this, it is recalled that in the regime of the steady intrusion any segment of the potential vorticity front moves in the negative \( y \) direction with the migration speed \( C \). The component of the velocity vector of the segment in the positive \( n \) direction (Fig. 8) is

\[
C_n = -C \cos \theta, \tag{4.24}
\]

which can be substituted into the path equation (4.1) to give

\[
A \frac{\partial^2 \theta}{\partial s^2} + C \cos \theta = 0. \tag{4.25}
\]

Since the asymptotic solution describes the flow only in the region \( 0 < x \sim L \), the boundary conditions (4.4) and (4.5) cannot be used and must be replaced by

\[
\theta \bigg|_{x = 0} = \pi/2 \tag{4.26}
\]

\[
\frac{\partial \theta}{\partial s} \bigg|_{x = \infty} = 0, \tag{4.27}
\]

meaning that the jet is straight and flows parallel to the wall at a large distance away from the separation point.

Equation (4.25) is identical to the nonlinear pendulum equation, which has the exact energy integral

\[
\frac{A}{2} \left( \frac{\partial \theta}{\partial s} \right)^2 + C \sin \theta = \text{const.} \tag{4.28}
\]

Substitution of the boundary conditions (4.17), (4.22), (4.26), and (4.27) into (4.28) yields

\[
\frac{(g')^2}{8fH(M_{10} + M_{20})}(H_1^2 - H_2^2)^2 + C \frac{M_{10} - M_{20}}{M_{10} + M_{20}} = C, \tag{4.29}
\]

which gives our previously derived solution (3.29), as should be the case.

f. Validity regime

It is natural to ask what is the actual criterion that determines the accuracy (and validity) of the developed as-
ymptotic theory. The crucial point in our derivation was "attaching" the free separated current to the colliding currents at the boundary of the near-wall region of strong interaction. In doing this, we have used the thin jet approximation, which is valid when the radius of curvature of the separated current is much larger than its width. It is not just the relative difference of the upstream near-wall depth of the countercurrent in the balanced and perturbed states that must be small. The near-wall curvature of the separated current must also be much less than the reciprocal Rossby radius in order to justify the use of the thin jet approximation. Using (4.17), we can express the validity criterion of the theory as

\[
\frac{(g')^{3/2}H^{1/2}(H_1^2 - H_2^2)}{2f(M_{10} + M_{20})} \ll 1. \tag{4.30}
\]

In a companion article (Lebedev and Nof 1996, manuscript submitted to Deep-Sea Res.), we describe complete numerical simulations of the shallow-water equations, which agree with our theory even if the above parameter is as large as 0.47.

5. Application to the South Atlantic Ocean

In this section we shall discuss an application of the model to the Brazil and Malvinas Currents (Fig. 1). It is recalled that we associate the model main current with the Brazil Current, whereas the Malvinas Current is represented by the model countercurrent. It is logical to associate the model stationary "balanced" currents with the long-term mean flow in the confluence region. First we shall choose the values of the undisturbed depth \((H)\) of the main current and the equilibrium near-wall depth \((H_0)\) that (more or less) fit the actual hydrographic measurements. Then we shall consider a physically reasonable perturbation of the balanced state and use the theory to deduce the resulting migration of the confluence zone.

There is a broad scatter in the estimates of the transports for both the Brazil and the Malvinas Currents in the oceanographic literature. For instance, the estimates of the Brazil Current transport range from 6.8 Sv \((= 10^6 \text{ m}^3 \text{s}^{-1})\) (Evans et al. 1983) to 70 Sv (Peterson 1990). To be consistent with our purely baroclinic model, we shall rely on the estimates relative to the intermediate reference levels (1000–2000 m), which yield a Brazil Current transport of 20 Sv (Gordon and Greengrove 1986; Stramma 1989). A reasonable value for the undisturbed depth of the main current is about 800 m, which yields a Rossby radius of 28 km for \(g' = 0.01 \text{ m s}^{-2}\) and \(f = 0.0001 \text{ s}^{-1}\). This model Rossby
radius compares favorably with the estimate of Garzoli and Bianchi (1987) derived from the mean Brunt-Väisälä frequency profile in the Brazil—Malvinas confluence area and the vertical normal mode decomposition (31 km).

Using the geostrophic transport formula, one finds that the main current's transport of 20 Sv, together with its undisturbed offshore depth of 800 m, correspond to a balanced near-wall depth ($H_0$) of approximately 500 m. Note that, due to the constraints of the balanced solution, this value uniquely determines the transport of the countercurrent, which turns out to be about 12 Sv. This qualitatively estimates the results of Gordon and Greengrove (1986) (10 Sv, 1400-m reference level) and Piola and Bianchi (1990) (10–12 Sv, 1000-m reference level). Other studies (e.g., Peterson 1992) indicate a strongly barotropic character of the Malvinas Current, which results in a total volume transport of up to 80 Sv. Clearly, such high transports cannot be used as a reference for our idealized 1/2-layer parameterization because our model does not include the barotropic component of the flow.

Given the simplified geometry and density structure of the model, it is hard to estimate the difference of the colliding current near-wall depths ($H_1 - H_2$) from observations. We believe, however, that a value of $O(0.1H) \sim 80$ m would be a reasonable guess. This corresponds to the sea surface height difference of about 0.8 m, qualitatively consistent with the satellite altimetry images of the area (Matano et al. 1993).

Consider now the perturbation of the described balanced state. A linear expansion of the formulas for the migration speed of a steadily propagating intrusion [(3.27) and (3.30)] with respect to the balanced depth $H_0$ yields the approximation

$$C \approx (g' H)^{1/2} \left\{ \begin{array}{ll} \left(H_0^* \right)^{1/2} (H_1^* - H_2^*)^2 & \quad \text{when } H_1^* > H_2^* \\
16H_0^* 2(2/3 - 2H_0^* 2 + 4H_0^* 3/3)^{-1} (H_2^* - H_1^*)^2 & \quad \text{when } H_1^* < H_2^*. \end{array} \right. $$

(5.1)

Substituting $H_0^* = (H_0/H) \sim 0.6$ and $H_1^* - H_2^* = 0.1$ into (5.1), we can see that $C \sim 2$ cm s$^{-1}$ when $H_1 > H_2$ (drift toward the Brazil Current) and $C \sim -4$ cm s$^{-1}$ when $H_1 < H_2$ (drift toward the Malvinas Current). These values appear to be in qualitative agreement with the observed rates of the countercurrent drift. According to Olson et al. (1988), the seasonal variations of the separation latitude can be as much as 400 km corresponding to a migration rate of about 3 cm s$^{-1}$ (for, say, 140 days).

Modeling the complex separation phenomenon with a simplified 1/2-layer f-plane inertial model is potentially problematic, even if it is fully nonlinear. It is natural to ask which of our results have a general physical foundation and which of them are specific to the given model. The theory is essentially based on two physical mechanisms: The first one is blocking of the main current path by the countercurrent so that both currents have no alternative but to veer offshore. Although in our model this is the only possibility, in reality the density of the cold Malvinas Current is greater than the density of the subtropical waters of the Brazil Current. Consequently, in principle, the Malvinas Current can, at least partially, dive under the Brazil Current and continue to flow along the coast under the surface. However, the observations indicate that this does not happen and that there is no alongshore "leakage" of the Malvinas current water beyond the separation latitude. Therefore, the parameterization of the actual density distribution by the reduced gravity model is adequate at least in the sense that it does not create a nonexisting effect.

The second physical mechanism is the inability of the straight geostrophically balanced separated current to tolerate variations of the transports of colliding currents. In our model, this causes curving of the separated current and initiates the migration of the separation point. Whether or not this feature is specific only for a 1/2-layer model and disappears in models with more realistic representation of the density field is not quite clear. An argument in support of the general character of that type of dynamics is that in the 21/2-layer model of separation considered by AN, the geostrophic transport of the separated current is also uniquely determined by the undisturbed upper-layer depth and cannot vary in order to compensate for changes of the colliding currents' transports.

A peculiar result of our study is that the main current transport is totally controlled by the countercurrent transport. It is not clear to what extent this interesting result is applicable to the real ocean. It has long been noticed that the transport of the Brazil Current is smaller than the value that follows from integrating the Sverdrup relation (Peterson and Stramona 1991). Possibly, the discrepancy can be attributed to the blocking of the Brazil Current transport by the collision with the Malvinas Current. It would be very interesting to reconcile this mechanism with the constraint imposed by the Sverdrup dynamics in the framework of a wind-driven model. However, even acknowledging that this type of control is perhaps unrealistic (because of viscosity, stratification, etc.) does not invalidate our model itself. In the framework of the model, the actual cause of the time-dependent separation is the nonzero difference between the sum of the colliding current trans-
ports \((T_1 + T_2)\) and the geostrophic transport of the separated current \((T_3)\).

An absence of \(\beta\) in our model raises another question. Can \(\beta\) prevent the migration mechanism that we have described? The \(\beta\)-plane generalization of the model is fairly straightforward. In fact, the path equation of CPR can be used since it was originally derived on a \(\beta\) plane. Also, the balances of mass and momentum in a near-wall control volume of \(O(R_d)\) are the same on both \(f\) and \(\beta\) planes, and the only required modification is to account for the alongshore variation of the near-wall depths of the colliding currents. Preliminary results have shown that, on a \(\beta\) plane, the system of colliding currents appears to be even more prone to the alongshore drift than on an \(f\) plane. This can be qualitatively understood from a typical near-wall depth distribution of the meridional currents, shown schematically in Fig. 12. The downstream decrease of the near-wall depth of both currents on a \(\beta\) plane follows from the consideration of the geostrophic transport. In the absence of the countercurrent, this effect leads to the separation due to vanishing upper-layer depth. When the stagnation point migrates from low to high near-wall depth, the background near-wall depth distribution gradually makes the depth difference on both sides of the stagnation point larger and accelerates the migration. This positive feedback may lead to an unlimited migration of the separation point even after the countercurrent transport is relaxed to its balanced value. To determine what bounds this nonphysical drift requires an investigation of a complete wind-driven system, which is beyond the scope of the present study.

In conclusion, the time-dependent collision and separation of inviscid boundary currents was examined in the framework of a nonlinear \(f\)-plane reduced gravity model. We were motivated by the idea that when the necessary conditions for the stationary separation are not satisfied, a nonstationary flow results. This can possibly explain the migration of the separation latitude observed in the South Atlantic.

We have found that the nonstationary collision and separation is caused by the inability of a straight geostrophic separated current to balance variable transports of colliding boundary currents. We have demonstrated that the evolution of the separated current can be described by the path equation of CPR with specially formulated boundary conditions. A consideration of the mass and momentum balances in a control volume (bounding the region of strong nonlinear interaction of colliding currents) allowed us to derive the required boundary conditions. As a result, the original nonlinear problem was reduced to the solution of a single path equation (Figs. 9, 10). We have proved the internal consistency of the path equation model by demonstrating analytically its agreement with the asymptotic solution for a steadily propagating intrusion. [A complete validation of the model by direct numerical simulation is reported in an accompanying article (Lebedev and Nof 1996, manuscript submitted to Deep-Sea Res.).] A qualitative application of the model to the Brazil and Malvinas Currents suggested that the imbalance of mass transports can be an important mechanism that causes observed migration of the separation latitude of these currents.

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**APPENDIX A**

**A Finite-Difference Scheme for the Path Equation**

The reader is referred to CPR for a proof that path equation (4.1) can be represented as

\[
\frac{\partial \theta}{\partial t} = A \frac{\partial^3 \theta}{\partial s^3} + \frac{1}{2} \left( \frac{\partial \theta}{\partial s} \right)^2 + C_0(t) \frac{\partial \theta}{\partial s},
\]

(A.1)

where the function \(C_0(t)\) is determined by the boundary conditions. Following the steps outlined in CPR, one can show that if one end of the current path is allowed to move along the wall, then

\[
C_0(t) = \sin \theta_0 \frac{dY_0}{dt} - \frac{A}{2} \left[ \frac{\partial \theta}{\partial s} (0, t) \right]^2.
\]

(A.2)

where \(Y_0(t)\) is the coordinate of the separation point. The value of \(dY_0/dt\) (i.e., the migration speed of the separation point) can be obtained by evaluating the path equation (4.1) at the wall \((s = 0)\),

\[
\frac{dY_0}{dt} = \frac{A}{\cos \theta_0} \left. \frac{\partial s^2 \theta}{\partial s^2} \right|_{s=0}.
\]

(A.3)

Equations (A.1)–(A.3) can be solved numerically with centered differences in space and a leapfrog
scheme in time. At any time (step \( n \)) we first compute the migration speed \((A.3)\) as

\[
\frac{dY_0}{dt} = \frac{2A}{\cos \theta_0 + \cos \theta_1} \Delta s \left[ \frac{\theta_0 - \theta_1}{2\Delta s} \left( \frac{\partial \theta}{\partial s} \right) \right]_0^n,
\]

\((A.4)\)

where \(\cos \theta_0\) and \(\left( \frac{\partial \theta}{\partial s} \right)_0\) are known from the boundary conditions. Substitution of \((A.4)\) into \((A.2)\) gives the value of \(C_0(t^*)\), which is used to make the time step in \((A.1)\). Before proceeding to the next time step, we use the leapfrog to advance the separation point according to \((A.4)\). The shape of the separated current is obtained from the geometric relations:

\[
X(s) = \int_0^s \cos \theta(\xi) d\xi
\]

\[
Y(s) = Y_0 + \int_0^s \sin \theta(\xi) d\xi \quad (A.5)
\]

by trapezoidal integration. In these formulas, \(X(s)\) and \(Y(s)\) are the Cartesian coordinates of a point situated a distance \(s\) (measured along the curve) from the wall.

In practice, the use of the boundary conditions \((4.4)\) and \((4.5)\) at \(s \to \infty\) (i.e., undisturbed jet at infinity) is inconvenient because they require a sufficiently large domain so that during the time integration the disturbances would not reach the boundary point. We have found that the “free tail” conditions

\[
\theta_{n-2} = \theta_N \quad (A.6)
\]

can be used as a possible simple type of open boundary condition. We have used them for all our multyyear computations.

**APPENDIX B**

**List of Symbols**

\(A\)  Coefficient in the path equation for the separated current

\(B\)  Bernoulli function along the wall

\(C\)  Speed of the steadily propagating intrusion

\(C^*\)  Nondimensional speed of the steadily propagating intrusion

\(C_n\)  Normal velocity of the potential vorticity front

\(D_{1,2}\)  Nondimensional near-wall depths of the main current and the countercurrent, respectively

\(f\)  Coriolis parameter

\(g'\)  “Reduced gravity,” \((\Delta \rho/\rho)g\)

\(H\)  Undisturbed depth of the main current

\(h\)  Depth (thickness of the upper layer)

\(h^*\)  Nondimensional depth

\(H_0^*\)  Nondimensional near-wall depth in the balanced state

\(H_{1,2}^*\)  Nondimensional near-wall depth of the main current and the countercurrent, respectively

\(h_{1,2,3}^*\)  Nondimensional depth distributions across the main current, the countercurrent, and the separated current, respectively

\(H_3^*\)  Nondimensional depth at the potential vorticity front

\(H_0\)  Near-wall depth in the balanced state

\(h_0(x, y)\)  Depth distribution for the balanced stationary flow

\(H_{1,2}\)  Near-wall depth of the main current and the countercurrent, respectively

\(h_{1,2,3}\)  Depth distributions across the main current, the countercurrent, and the separated current, respectively

\(H_3\)  Depth at the potential vorticity front

\(K_0\)  Nondimensional curvature of the potential vorticity front

\(L\)  Width of the steadily propagating intrusion (also, “collision length scale”)

\(M_{20}^*\)  Nondimensional flux of the countercurrent in the balanced state

\(M_{1,2}\)  Momentum flux of the main current and the countercurrent, respectively

\(M_{10,20}\)  Momentum fluxes of colliding currents in the balanced state

\((n, s)\)  Curvilinear coordinates associated with the potential vorticity front

\(R_d\)  Rossby deformation radius, \((g' H)^{1/2}/f\)

\(s\)  Distance measured along the potential vorticity front

\(T_{1,2,3}\)  Volume transports of the main current, countercurrent, and separated current, respectively

\(u_0(x, y),\ v_0(x, y)\)  Velocity field for a stationary balanced flow

\((u^*, v^*)\)  Nondimensional Cartesian velocity components

\(V\)  Near-wall velocity

\(V_0, V_{1,2}, V_3\)  Nondimensional near-wall velocity in the balanced state

\(V_{1,2}\)  Nondimensional velocity at the potential vorticity front

\(\varepsilon\)  Near-wall velocity in the balanced state

\(\varepsilon\)  Near-wall velocity of the main current and countercurrent, respectively

\(\gamma_0\)  Velocity at the potential vorticity front

\(\gamma_0^*\)  Small nondimensional parameter \([H_1 - H_0]/H]\)

\(\gamma\)  Nondimensional width of the countercurrent in the balanced state

\(\gamma_0\)  Width of the countercurrent in the balanced state
\[ \gamma_2 \]
\[ \gamma_s \]
Width of the countercurrent
Distance from the potential vorticity front to the surancing interface
\[ \theta \]
Separation angle
\[ \theta_0 \]
Separation angle of the balanced currents
\[ \tau \]
Timescale of the nonstationary separation
\[ \psi \]
Streamfunction
\[ \psi^* \]
Nondimensional streamfunction

REFERENCES