On the β-Induced Movement of Isolated Baroclinic Eddies

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ABSTRACT

In this paper an analytical method is proposed for calculating the nonlinear β-induced translation of isolated baroclinic eddies. The study focuses on frictionless anticyclonic eddies with a uniform anomalous density and a lens-like cross section which translates steadily in a resting ocean. The depth of these eddies vanishes along the outer edge so that as they translate westward their entire mass anomaly is carried along with them.

The proposed method for calculating the translation speed incorporates the nonlinear equations of motion in an integrated form and a simple perturbation scheme. It relates the translation of the eddy to its intensity, size and volume, but requires only an approximate knowledge of the corresponding numerical values.

The power and usefulness of the proposed method is demonstrated by its application to a class of simply-structured eddies whose swirl velocity increases monotonically with the distance from the center. It is found that the translation of these eddies is considerably smaller than that of a simple Rossby wave. A small Rossby number eddy whose swirl velocity increases monotonically with the distance from the center translates westward at approximately $\frac{1}{3}\beta R_d^2$ (where R_d is the deformation radius), whereas the most nonlinear eddy (whose negative relative vorticity approaches the vorticity of the earth) translates at $\frac{2}{3}\beta R_d^2$.

The proposed method is tested by its application to more complicated anticyclonic eddies representing those shed by the Loop Current in the Gulf of Mexico. For these eddies, the predicted westward translation speed is $0.32\beta R_d^2$ which agrees very well with both numerical experiments and field observations.

1. Introduction

It has been recognized for some time that mid-ocean eddies play an important role in ocean dynamics and in affecting the structure of the ocean. Of particular importance is the question of their translation because it is directly related to their influence on the distribution of energy and properties within the ocean.

In the present study, we shall examine analytically the nonlinear translation of anticyclonic baroclinic eddies with a uniform density anomaly. We shall consider isolated eddies whose depth vanishes along their outer edge corresponding to a lens-like cross section (Fig. 1). Various observations show that the basic structure of a large number of eddies belongs to this category. Examples are the eddies shed by the Loop Current in the Gulf of Mexico (Elliott, 1979), Amazonian eddies (see, e.g., Ryther *et al.*, 1967; Nof, 1981), and the Mediterranean eddies observed off the Bahamas (McDowell and Rossby, 1978).

A considerable number of both analytical and numerical investigations have previously addressed various aspects of the β-induced movement of planetary eddies. Among the former studies are those of Rossby (1948), Warren (1967), Firing and Beardsley (1976), Larichev and Reznik (1976), Flierl (1977), Flierl *et al.* (1980), Flierl (1979) and Shen (1981). The analytical studies are mostly constrained to linear or quasi-linear dynamics and, while being informative, they do not deal directly with the isolated lens-like eddies considered in this paper. The numerical models of McWilliams and Flierl (1979), Meid and Lindemann (1979) and Hulburt and Thompson (1980) do not specifically deal with our problem either, but some of them do shed light on some aspects of the problem as we shall see later.

Our attention will be focused on frictionless anticyclonic eddies which, due to their lens-like shape, correspond to an edge with a discontinuity in the density, potential vorticity and possibly the velocity (see Fig. 1). The influence of steering currents will be neglected and it will be assumed that the eddies are entirely isolated in the sense that they are free from any interactions with boundaries, mean flow or other eddies. Steadily translating states corresponding to both the mass and momentum being carried with the eddy will be sought.

To obtain the solution for the problem, the equa-

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tions of motion are integrated over the whole eddy in a coordinate system moving with the eddy itself. The resulting equations are then combined with a perturbation scheme in $\epsilon = \beta l/f_0$, the ratio between the variation of the Coriolis parameter across the eddy to the Coriolis parameter in the center, and give a simple relationship for the translation speed.

This simple relationship enables one to compute the translation of both linear and nonlinear eddies provided that their approximate structure is known from either observations or theory. To illustrate the power of the derived expression it is applied to eddies with various structures corresponding to various translation speeds. It is applied first to a class of simple anticyclonic eddies whose swirl velocity increases linearly with the distance from the center. As we shall see, calculation of the β-induced translation does not require knowledge of the exact eddy structure and shape. Rather, it is sufficient to know the structure that the eddy would have in the absence of β (i.e., on an f plane). Using this information, the translation speed is calculated for the whole possible range of Rossby numbers. The computed values of the translation speed indicate that the presence of nonlinearity increases the westward movement significantly. However, the translation is always smaller than βR_d^2 (where R_d is the deformation radius) even for the most nonlinear eddies.

After presenting the behavior of these relatively simple eddies, more complicated eddies, which can be thought of as representing the anticyclonic eddies in the Gulf of Mexico, are considered. The translation speed is calculated using the same general method; the predicted results are then compared to the numerical experiments of Hulburt and Thompson (1980) and to the field observations discussed by Elliott (1979). It is found that the discrepancy between the model prediction and both the numerical experiments and the field observations is <20%.

This paper is organized as follows: The formulation of the problem is presented in Section 2 and the perturbation analysis in Section 3. The general theory is then applied to simply structured eddies (Section 4) and to more complicated eddies corresponding to those observed in the Gulf of Mexico (Section 5). The results of the study are summarized in Section 6.

2. Formulation

Consider the two-layer model shown in Fig. 1. As mentioned earlier, the eddy is embedded in a resting ocean and the lower layer is assumed to be infinitely deep. We shall seek solutions corresponding to an eddy which translates zonally at a constant speed C (positive for eastward motion and negative for westward). It is assumed that the eddy shape

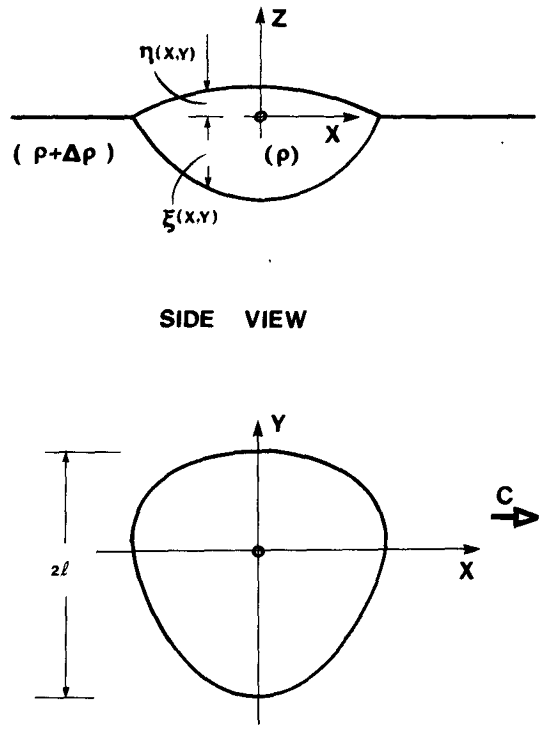

FIG. 1. Schematic diagram of the model under study. The anticyclonic eddy has a lens-like cross section and is bounded by a free streamline. The free surface vertical displacement $\eta(x, y)$ and the interface displacement $\xi(xy)$ are measured upward and downward, respectively, from the undisturbed level of the surrounding fluid. The eddy is translating zonally at a constant speed C (positive for eastward movement and negative for westward).

does not change much in time so that in a coordinate system moving with the eddy itself the motion can be taken to be steady. It is not *a priori* obvious under what conditions this assumption is adequate, but using scaling arguments it will be demonstrated later that the assumption is valid as long as $(\beta l/f_0)^2 \ll$ Ro, where l is the eddy size defined as half the distance between the northernmost and southernmost edges, and Ro is the Rossby number. As we shall see, this condition is satisfied by many eddies of practical interest.

The origin of our moving coordinates system is located at the center of the eddy; the x and y axes are directed toward the east and north, respectively. The equations of motion and continuity for this moving coordinates system are obtained by applying the transformations $x' \rightarrow x + ct$, $y' \rightarrow y$ (where x' and y' are the fixed stationary coordinates). For hydrostatic motions, the governing equations are found to be

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - (f_0 + \beta y)v = -g'\frac{\partial h}{\partial x}, \quad (2.1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (f_0 + \beta y)u + f_0 C = -g'\frac{\partial h}{\partial y}, \quad (2.2)$$

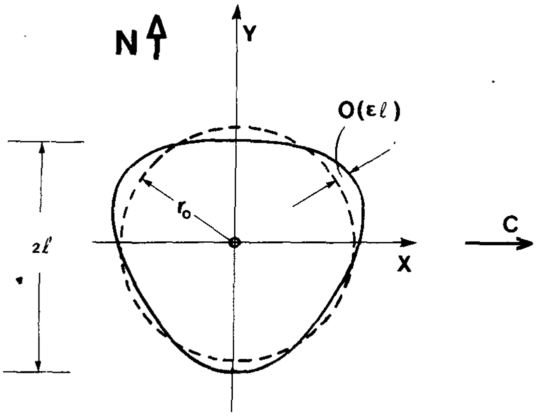

FIG. 2. Schematic diagram of the shape distortion imposed by the presence of β. The distorted shape (solid line) is symmetrical with respect to the x axis but asymmetrical with respect to y. The deviation of the outermost streamline from an exact circle (dashed line) is of O(ϵl).

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{2.3}$$

Here u and v are the depth-independent [$u = u(x,y)$; $v = v(x,y)$] horizontal velocity components, and h is the total depth (i.e., $h = \xi + \eta \approx \xi$, where $\eta(x,y)$ is the free surface displacement which is measured upward from the origin and $\xi(x,y)$ is the interface displacement which is measured downward). The Coriolis parameter at $y = 0$ is denoted by f_0, the density difference between the two layers is $\Delta\rho$, and g' is the "reduced gravity" defined by $g' = (\Delta\rho/\rho)g$.

Note that the term f_0C on the left-hand side of (2.2) results from the fact that in a moving coordinate system there exists an additional force acting on all fluid parcels. In general, the left-hand side of (2.2) should also include the term βyC; however, this has been neglected because the translation speed itself is small so that βyC is much smaller than the other terms [since $(\beta l/f_0) \ll 1$]. As we shall see, this neglect is consistent with the perturbation scheme which will be used in Section 3 because terms of O($\beta l C$) are associated with the second-order approximation which is not included in the analysis.

It also should be noted that although the lower layer is not motionless in our moving coordinate system, the relationship $\eta = (\Delta\rho/\rho)\xi$ is valid and, therefore, the approximation $g\eta = g'h$, which has been used to obtain (2.1) and (2.2), is adequate. This results from the fact that the ratio between the eddy depth to the depth of the ambient fluid approaches zero; consequently, application of potential vorticity conservation to the lower layer implies that the speed of the lower layer is uniform and equals $-C$ everywhere. That is, since the eddy is thin and shallow it moves on top of the infinitely deep lower layer without altering the fluid underneath or around its edge. Hence, there are no pressure gradients in the lower layer and this requires $\eta = (\Delta\rho/\rho)\xi$. It is important to realize, however, that the situation would have been quite different had the lower layer not been infinitely deep. With a finite lower layer there will probably be important interactions between the eddy and its surroundings so that the right-hand side of (2.1) and (2.2) cannot be expressed in terms of g' and h alone.

The system (2.1)–(2.3) is subject to the boundary conditions

$$\left.\begin{aligned} h = 0,\ \phi(x,y,f_0,\beta,g',C,l) = 0 \\ (\mathbf{i}u + \mathbf{j}v)\cdot\nabla\phi = 0,\ \phi(x,y,f_0,\beta,g',C,l) = 0 \end{aligned}\right\},$$

where ∇ is the horizontal del operator. The first condition states that $h = 0$ along a curve which is not known in advance (ϕ) while the second requires that this curve be a streamline. These conditions correspond to the fact that the location and shape of the eddy outer edge are not known *a priori*, but rather must be determined as part of the problem. Note that since our model is inviscid the velocity can be discontinuous across ϕ.

Before proceeding any further, we shall analyze and discuss the symmetrical and asymmetrical properties of (2.1)–(2.3). To do so (2.1)–(2.3) are manipulated to give the potential vorticity equation and Bernoulli integral:

$$\nabla\cdot\left(\frac{\nabla\psi}{h}\right) + f_0 + \beta y = hK(\psi), \tag{2.4}$$

$$\frac{1}{2}\left(\frac{\nabla\psi}{h}\right)^2 + g'h + f_0Cy = G(\psi), \tag{2.5}$$

where $K(\psi) = dG(\psi)/d\psi$, and ψ is a transport function defined by

$$\frac{\partial\psi}{\partial y} = -uh;\quad \frac{\partial\psi}{\partial x} = vh. \tag{2.6}$$

The functions $K(\psi)$ and $G(\psi)$ depend on the initial conditions to the problem, but knowledge of their detailed structure is not necessary for the present discussion.

If $\beta = 0$ and $C = 0$ (i.e., there is no translation and the eddy is stationary) the system (2.4)–(2.5) is invariant to the transformations $x \rightarrow -x$ and $y \rightarrow -y$. Hence, any closed streamlines associated with an eddy on an f plane ($\beta = 0$) will be symmetrical with respect to both the x and y axes. Furthermore, it is easy to show (using polar coordinates) that under these conditions, Eqs. (2.4)–(2.5) always possess solutions which are radially symmetric corresponding to purely circular motion.

On the other hand, when $\beta \neq 0$ and $C \neq 0$ the system is invariant to the transformation $x \rightarrow -x$ but variant to the transformation $y \rightarrow -y$. Consequently, a steadily translating eddy on a β plane will be symmetrical with respect to the x axis but asymmetrical with respect to the y axis. Due to the smallness of the perturbations imposed by the presence of β ($\beta l \ll f_0$) it is expected that the streamlines associated with a steadily translating eddy will correspond to slightly distorted circles symmetrical with respect to x but asymmetrical with respect to y as shown schematically in Fig. 2.

These theoretical considerations which suggest that the presence of β does not cause large deviations from exact circles are supported by a number of field observations. For instance, the structure of the Loop Current eddy shown in Fig. 3 corresponds to a slightly distorted circle as expected theoretically. However, one can see that the theoretically predicted asymmetry (resulting from the structure of the governing equations) cannot be clearly identified probably due to other processes such as local steering currents or wind action which distort the eddy shape.

Since the structure of a steadily translating eddy on a β plane is now qualitatively understood, we shall proceed and derive the general expression for the translation speed. Multiplication of (2.2) by h and integration over the whole eddy give

$$\iint_S \left(hu\frac{\partial v}{\partial x} + hv\frac{\partial v}{\partial y}\right)dxdy + \iint_S (f_0 + \beta y)uhdxdy + \iint_S f_0 Chdxdy = -\frac{g'}{2}\iint_S \frac{\partial}{\partial y}(h^2)dxdy, \quad (2.7)$$

where S denotes the entire area of the eddy. Using (2.3) and Stokes' theorem, Eq. (2.7) can be written as

$$-\int_\Phi hv^2dx + \int_\Phi huvdy + \iint_S (f_0 + \beta y)uhdxdy + \iint_S f_0 Chdxdy = \frac{g'}{2}\int_\Phi h^2dx, \quad (2.8)$$

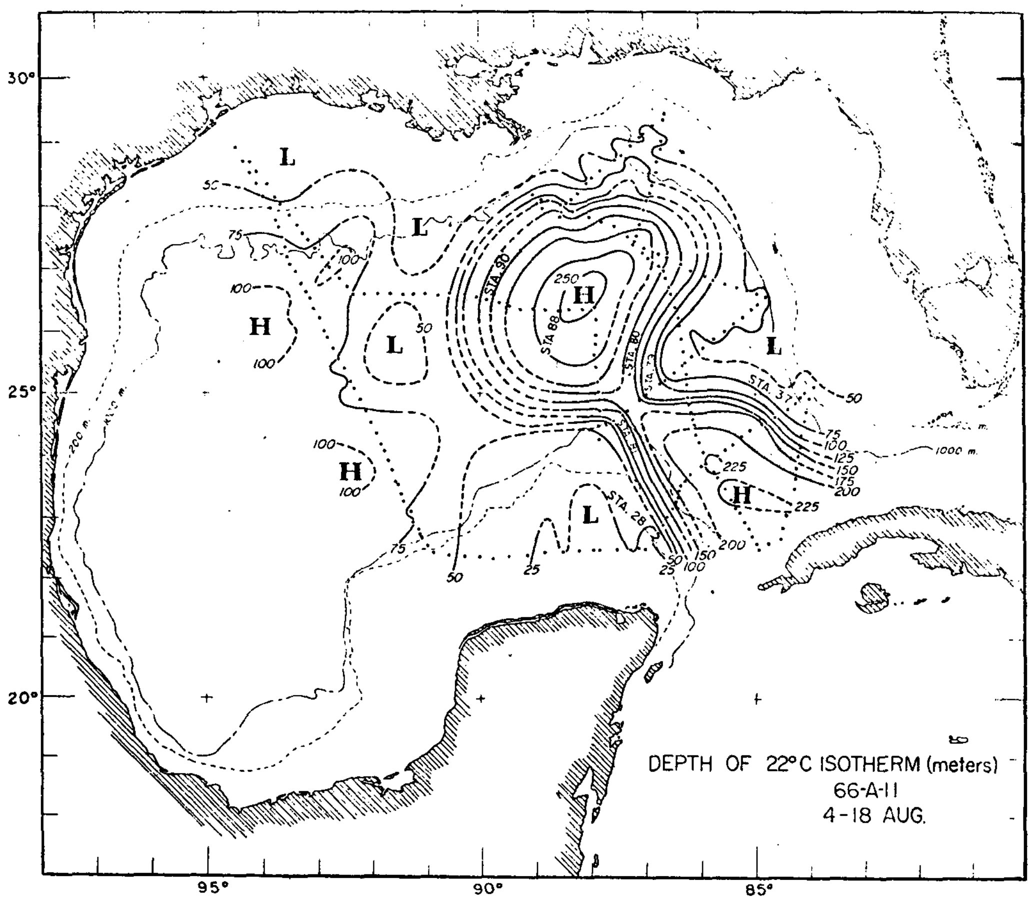

FIG. 3. The structure of a typical eddy in the Gulf of Mexico shortly before separation (reproduced from Leipper, 1970). The contours correspond to the topography of the 22°C isothermal surface as determined from data collected by the R/V *Alaminos* during October–November 1966. Note that the distortions of the contours from an exact circle are relatively small.

where, as mentioned earlier, ϕ denotes the free-bounding streamline representing the eddy edge. Since $h = 0$ along ϕ, the integrals over the nonlinear terms and the pressure term vanish identically so that (2.8) reduces to

$$\iint_S (f_0 + \beta y)uh dxdy + \iint_S f_0 Ch dxdy = 0. \quad (2.9)$$

In view of (2.6), Eq. (2.9) can be written in the form

$$-\iint_S \left\{\frac{\partial}{\partial y}[(f_0 + \beta y)\psi] - \beta\psi\right\}dxdy + f_0 C \iint_S h dxdy = 0,$$

which by defining ψ to be zero on the boundary (i.e., $\psi = 0$, along ϕ) can be further simplified to

$$C = -\beta \iint_S \psi dxdy \Big/ f_0 \iint_S h dxdy. \quad (2.10)$$

Eq. (2.10) represents a balance between the net force due to β and the force due to the translation. If $C = 0$, Eq. (2.10) reduces to $\iint \psi dxdy = 0$, which is equivalent to the condition used by Stern (1975) for a stationary barotropic eddy. For a single anticyclonic eddy whose depth vanishes along the edge, the integral $\iint \psi dxdy$ is always positive and cannot be equal to zero; hence, we conclude that the free eddy must always translate westward. Furthermore, since the denominator represents the volume of the eddy, we see that eddies corresponding to high swirl velocity will translate faster than those with low swirl velocity.

Eq. (2.10) suggests that knowledge of the approximate structure of ψ and h is sufficient for calculating the translation. To determine the specific necessary information, a perturbation scheme is applied in the next section.

3. Perturbation analysis

In the subsequent analysis the following nondimensional variables will be used:

$$\left.\begin{aligned} &x^* = x/l; \quad y^* = y/l; \quad u^* = u/\mathrm{Ro} f_0 l \\ &v^* = v/\mathrm{Ro} f_0 l \\ &\psi^* = \psi/\mathrm{Ro} f_0 l^2 \hat{h}; \quad R_d = (g'\hat{h})^{1/2}/f_0 \\ &\epsilon = \beta l/f_0 \\ &h^* = h/\hat{h}; \quad C^* = C/\mathrm{Ro} f_0 l \end{aligned}\right\} . \quad (3.1)$$

Here l is the eddy size (defined earlier in Section 2), $\hat{h}$ the maximum eddy depth (at the center where $x = y = 0$), R_d the internal deformation radius, Ro the Rossby number (Ro $\lesssim$ 1) and ϵ ($\ll$1) represents the ratio between the variation of the Coriolis parameter across the eddy to the Coriolis parameter at the center (f_0). In terms of these nondimensional variables, Eq. (2.10) is

$$C^* = -\epsilon \iint_{S^*} \psi^* dx^* dy \Big/ \iint_{S^*} h^* dx^* dy^*, \quad (3.2)$$

where S^* is the nondimensional area of the eddy ($S/\pi l^2$).

It is further assumed that the dependent variables posses power series expansions in ϵ, e.g.,

$$\psi^*(x^*, y^*, \epsilon) = \psi^{(0)}(x^*, y^*) + \epsilon\psi^{(1)}(x^*, y^*) + \cdots, \quad (3.3a)$$

$$h^*(x^*, y^*, \epsilon) = h^{(0)}(x^*, y^*) + \epsilon h^{(1)}(x^*, y^*) + \cdots, \quad (3.3b)$$

which state that the structure of a steadily translating eddy on a β plane does not differ much from the structure that the eddy would have on an f plane.

Similarly, the translation speed is expanded as

$$C^* = \epsilon C^{(1)} + \epsilon^2 C^{(2)} + \cdots. \quad (3.4)$$

By substituting (3.3) into (3.2), considering (3.4), and noting that any function A satisfies

$$\iint_{S^*} A dx^* dy^* = \iint_{S^{(0)}} A dx^* dy^* + O(\epsilon),$$

[where $S^{(0)}$ is the area that the eddy would have in the absence of β], one finds

$$C^* = -\epsilon \iint_{S^{(0)}} \psi^{(0)} dx^* dy^* \Big/ \iint_{S^{(0)}} h^{(0)} dx^* dy^* + O(\epsilon^2). \quad (3.5)$$

Eq. (3.5) relates the translation speed [up to $O(\epsilon^2)$] to the basic zeroth-order state. The fact that the right-hand side of (3.5) includes only zeroth-order terms simply means that the influence of the shape distortion and the perturbed velocities on the translation speed is of $O(\epsilon^2)$. It shows that in order to calculate the translation of a given eddy it is sufficient to know the structure that the eddy would have on an f plane.

Since the zeroth-order state is always radially symmetric, it is convenient to express (3.5) in polar variables; for clarity we shall subsequently use the polar variables in dimensional form. In these variables, (3.5) is

$$C = -\beta \int_0^{r_0} \int_{r_0}^{r} \bar{V}_\theta(r)\bar{h}(r) dr r dr \Big/ f_0 \int_0^{r_0} \bar{h}(r) r dr + O\left[\left(\frac{\beta r_0}{f_0}\right)^2 \bar{V}_\theta\right], \quad (3.6)$$

where $\bar{V}_\theta(r)$ and $\bar{h}(r)$ are the dimensional swirl velocity and depth corresponding to the basic state ($\beta = 0$) and r_0 is the dimensional radius of the eddy edge corresponding to the same state [$\bar{h}(r_0) = 0$]. For linear eddies (Ro $\ll$ 1), relation (3.6) can be further simplified to

$$C = -\beta g' \int_0^{r_0} [\bar{h}(r)]^2 r dr \Big/ 2f_0^2 \int_0^{r_0} \bar{h}(r) r dr + O\left[\left(\frac{\beta r_0}{f_0}\right)^2 \bar{V}_\theta\right] + O\left[\mathrm{Ro}\,\frac{\beta r_0}{f_0}\bar{V}_\theta\right]. \quad (3.7)$$

As pointed out earlier, our model is valid as long as the eddy translates at constant speed without changing its shape and structure with time. Since our analysis does not include the complete detailed solution of the first-order problem we cannot unequivocally answer the question of under what conditions the assumption is adequate. Nevertheless, using scaling arguments it will be shown below that even if the eddy shape does change in time, the changes are small and negligible for many free eddies.

To show this, it is recalled that due to the presence of β the streamlines correspond to asymmetrical slightly distorted circles (Fig. 2). The corresponding horizontal distortion of the outermost streamline is of $O(\epsilon l)$ and if the shape changes with time due to β, then it is associated with a time scale of $O(1/\beta l)$. Hence, the change in time of the eddy shape corresponds to a speed of $O(\epsilon\beta l^2)$. That is to say, if the total velocity is written as $u(x, y, \epsilon, t) = \bar{u}(x, y, \epsilon) + u'(x, y, \epsilon, t)$, then the time-dependent component $u'(x, y, \epsilon, t)$ is of $O(\epsilon\beta l^2)$.

Eq. (2.2) shows that the neglect of time-dependent motions is justified as long as $\partial u/\partial t \ll f_0 C$. By taking this into account, considering $C \sim O(\beta l\bar{u}/f_0)$, $t \sim O(1/\beta l)$, and the order of magnitude of u', one finds that the neglect is justified as long as Ro $\gg (\beta l/f_0)^2$. This condition is satisfied by many eddies because the parameter $(\beta l/f_0)^2$ is typically of $O(10^{-3})$, whereas the Rossby number is of O(0.1). We therefore see that our assumption of steadily translating eddies is probably adequate for a large number of eddies.

4. Simple linear and nonlinear eddies

In this section we shall analyse the translation associated with a class of simply structured eddies. With the aid of these presentations it will be demonstrated how the translation speed can be computed and the differences between linear and nonlinear eddies will be examined. As we shall see, these simple eddies may be unstable on both f and β planes. Nevertheless, they provide a rather clear way of illustrating the fundamental points under discussion and it is instructive to examine them before considering more complicated structures.

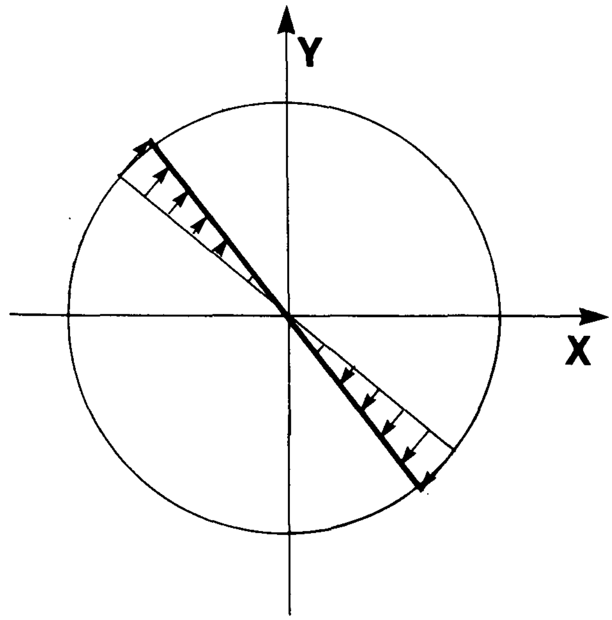

FIG. 4. The adopted f-plane velocity structure for an anticyclonic eddy [$\bar{v}_\theta = -\mathrm{Ro} f_0 r$].

Consider an eddy whose radially symmetric structure on an f plane corresponds to a linear increase of the tangential velocity as shown in Fig. 4. For the purpose of the present discussion it is not important to know how (or why) an eddy with such a structure was formed. The reader who is interested in this aspect is referred to Nof (1981) where the formation of anticyclonic eddies whose constant relative vorticity equals $-f_0$ is discussed.

The velocity distribution corresponding to Fig. 4 is

$$\bar{v}_\theta = -\mathrm{Ro} f_0 r, \quad (4.1)$$

where the bar denotes that the variable in question is associated with the basic state ($\epsilon = 0$). The depth of the eddy is found by integrating the nonlinear momentum equation

$$\frac{\bar{v}_\theta^2}{r} + f_0\bar{v}_\theta = g'\frac{d\bar{h}}{dr}, \quad (4.2)$$

which gives

$$\bar{h}(r) = \hat{h} + \mathrm{Ro} f_0^2 r^2(\mathrm{Ro} - 1)/2g', \quad (4.3)$$

where, as used previously, $\hat{h}$ denotes the maximum depth (at $r = 0$). Eq. (4.3) shows that the interface strikes the free surface ($\bar{h} = 0$) along the circle

$$r_0 = f_0^{-1}\left[\frac{2g'\hat{h}}{\mathrm{Ro}(1 - \mathrm{Ro})}\right]^{1/2}, \quad (4.4)$$

indicating that for a given $\hat{h}$, a linear eddy (Ro $\ll$ 1) will be larger than a nonlinear eddy [(Ro $\sim$ O(1)].

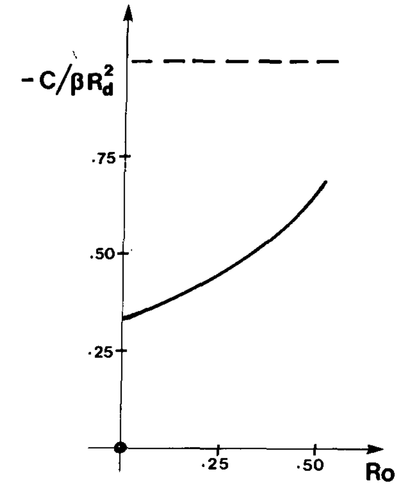

FIG. 5. The β-induced translation of an anticyclonic eddy [whose f plane structure corresponds to a linear velocity distribution (Fig. 4)] as a function of the Rossby number (solid line). Note that here the translation speed is expressed in terms of the deformation radius which is by itself a function of Ro (see text). Hence, the graph does not correspond to eddies with identical volumes or identical sizes. The dashed line represents the speed of a Rossby wave (βR_d^2).

Note that there is an upper bound to the degree of nonlinearity because an anticyclonic eddy cannot have relative vorticity $[(1/r)d(r\bar{v}_\theta)/dr]$ larger than f_0. Consequently, the largest Rossby number which can be associated with the eddy is 1/2.

The β-induced translation is now found, by substituting (4.1), (4.3) and (4.4) into (3.6), to be

$$C = -\tfrac{1}{3}\beta R_d^2(1 - \mathrm{Ro})^{-1} + O\left(\frac{\beta^2 r_0 R_d^2}{f_0}\right), \quad (4.5)$$

which in terms of the eddy radius is

$$C \approx -\tfrac{1}{6}\beta\, \mathrm{Ro}\, r_0^2. \quad (4.6)$$

Since the Rossby number is related to the ratio between the deformation radius R_d and the eddy radius r_0 [see (4.4)], the translation speed also can be expressed in the form

$$C \approx -\tfrac{2}{3}\beta R_d^2\{1 + [1 - 8(R_d/r_0)^2]^{1/2}\}^{-1}. \quad (4.7)$$

which for a Rossby number that is small but not entirely negligible [i.e., $(R_d/r_0)^2 < 1$] reduces to

$$C = -\tfrac{1}{3}\beta R_d^2\left[1 + 2\left(\frac{R_d}{r_0}\right)^2\right] + O\left[\frac{\beta R_d^2}{3}\left(\frac{R_d}{r_0}\right)\right]^4 + O\left(\frac{\beta^2 r_0 R_d^2}{f_0}\right). \quad (4.8)$$

This relationship indicates that the translation of our eddy is about one-third of the speed of a baroclinic solitary wave on a β plane (Flierl, 1979).

The dependence of the translation speed on the Rossby number [as given by (4.5)] is shown in Fig. 5. We see that for linear eddies (Ro $\ll$ 1) the translation is $-\tfrac{1}{3}\beta R_d^2$, whereas for the most nonlinear eddy (Ro = ½) the translation is $-\tfrac{2}{3}\beta R_d^2$. Thus, a nonlinear eddy translates twice as fast as a linear eddy which has the same deformation radius. In a sense, the effect of nonlinearity is even larger because nonlinear eddies tend to be deeper than linear eddies and, hence, they have larger deformation radii. To illustrate this point the maximum depth $\hat{h}$ and the translation speed C (4.5) are expressed in terms of the eddy volume (Q). From (4.4) and (4.5) one finds that, in terms of Q, the maximum depth and the translation speed are

$$\hat{h} = f_0\left[\frac{\mathrm{Ro}(1 - \mathrm{Ro})Q}{\pi g'}\right]^{1/2}, \quad (4.9)$$

$$C = -\frac{\beta}{3f_0}\left[\frac{g'\,\mathrm{Ro}Q}{\pi(1 - R_0)}\right]^{1/2}. \quad (4.10)$$

Relations (4.9) and (4.10) are shown in Figs. 6 and 7, respectively. Fig. 6 shows that if the volume Q is held fixed and the Rossby number is increased (by increasing the swirl speed, for example) from, say, 0.05 to 0.50, the nondimensional maximum depth $[\hat{h}f_0^{-1}(Q/\pi g')^{-1/2}]$ increases from 0.22 to 0.50. For the same increase in Ro, the nondimensional translation speed $[C(\beta/3f_0)^{-1}(g'Q/\pi)^{-1/2}]$ increases from 0.23 to 1.00. The latter corresponds to an increase of ~400% which is much larger than the maximum increase associated with eddies whose deformation radii are identical. These considerations illustrate that the total effect of nonlinearity is actually larger than that indicated by Fig. 5.

The physical reality of the analysis presented in this section depends, of course, on the stability of the eddies. A detailed examination of the stability question is beyond the scope of this study, but some aspects of the problem can be examined by evaluat-

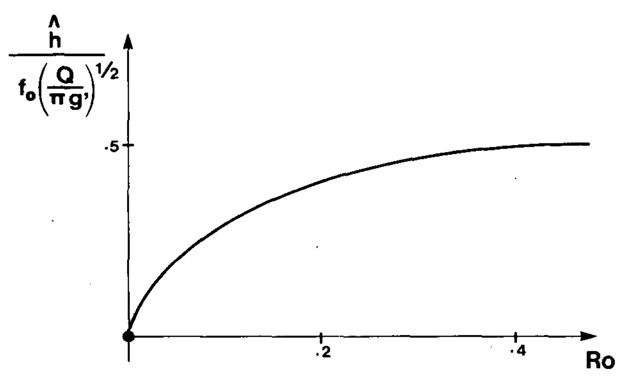

FIG. 6. The nondimensional maximum depth of an anticyclonic f-plane eddy with a fixed volume and a linear velocity distribution (Fig. 4), as a function of the Rossby number.

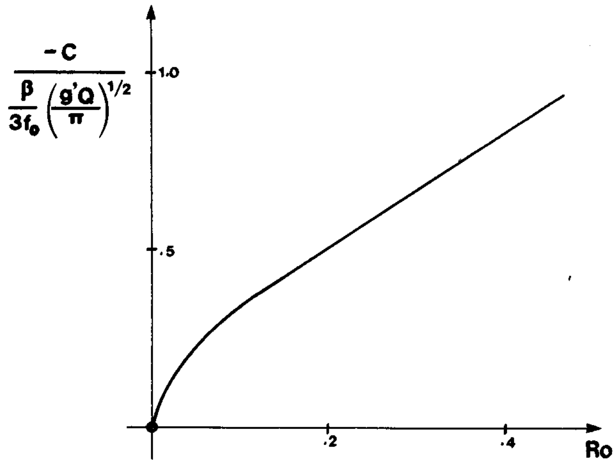

FIG. 7. The nondimensional β-induced translation of an anticyclonic eddy (whose f plane structure correspond to Fig. 4) as a function of the Rossby number (solid line). Here, the volume of the eddy Q is kept fixed while the depth $\hat{h}$ (and, consequently, the deformation radius) varies with the Rossby number. Note that the effect of nonlinearity (high Rossby number) is more pronounced than in Fig. 5 where the volume is not fixed.

ing the magnitude of the Richardson number ($\mathrm{Ri} = g'h/\bar{v}_\theta^2$). Using the structure of the eddy in the absence of β, one finds that the Richardson number is smaller than a quarter in the ring, i.e.,

$$r_0\left(\frac{1-\mathrm{Ro}}{1-\frac{1}{2}\mathrm{Ro}}\right)^{1/2} \leqslant r < r_0; \quad 0 \leqslant \theta \leqslant 2\pi.$$

Hence, for $\mathrm{Ro} \rightarrow 0$ the region of possible local instability is confined to the edge itself ($r = r_0$), but for $\mathrm{Ro} = \frac{1}{2}$ the region is between $0.87r_0$ and r_0. As we shall see in the next section, the question of stability around the eddy's edge is resolved if we choose an eddy whose velocity vanishes as $h \rightarrow 0$. It should be pointed out, however, that even if $\bar{v}_\theta \rightarrow 0$ as $h \rightarrow 0$, other processes such as baroclinic instability may be active in the eddy. The laboratory experiments of Griffiths and Linden (1981) for eddies with constant potential vorticity suggest that some eddies are baroclinically unstable.

5. Loop Current eddies

In this section the proposed method of calculating the translation speed will be tested by its application to the anticyclonic eddies shed by the Loop Current in the Gulf of Mexico. These eddies have been observed to break off from the Loop Current and translate into the Western Gulf. Fig. 3 which shows the structure of such an eddy indicates that their length scale is ~180 km.

In contrast to the previous section, where eddies whose f-plane structure corresponds to a linear velocity distribution have been considered, we shall adopt here an eddy whose f-plane structure corresponds to a parabolic velocity distribution. The adopted velocity structure is given by

$$\bar{v}_\theta = 2\,\mathrm{Ro} f_0 r\left(\frac{r}{r_0} - 1\right) \tag{5.1}$$

and is shown in Fig. 8. As indicated previously, the details of the process (or processes) which led to such an eddy are not important for the present analysis.

The velocity distribution given by (5.1) is more "realistic" than the previously considered linear distribution for two reasons. First, most eddies tend to have a region of maximum speed associated with zero velocity at the center of the eddy and vanishing velocity at some distance r. Second, as we shall shortly see, an eddy associated with a parabolic velocity distribution does not have a Richardson number smaller than ¼ near its edge.

Note that, as before, there is an upper bound on the magnitude of the Rossby number which can be associated with the velocity structure (5.1). It cannot assume values larger than ¼ because otherwise the negative relative vorticity would be larger than f_0 (as $r \rightarrow 0$) which is impossible. For the newly adopted velocity structure (5.1), the depth distribution is found from (4.2) to be

$$\bar{h}(r) = \hat{h} + \mathrm{Ro} f_0^2 r^2(2\,\mathrm{Ro} - 1)/g' + 2\,\mathrm{Ro} f_0^2 r^3(1 - 4\,\mathrm{Ro})/3g'r_0 + \mathrm{Ro}^2 f_0^2 r^4/g'r_0^2. \tag{5.2}$$

The corresponding eddy radius (r_0) is

$$r_0 = \left[\frac{3g'\hat{h}}{\mathrm{Ro}(1-\mathrm{Ro})}\right]^{1/2} f_0^{-1}. \tag{5.3}$$

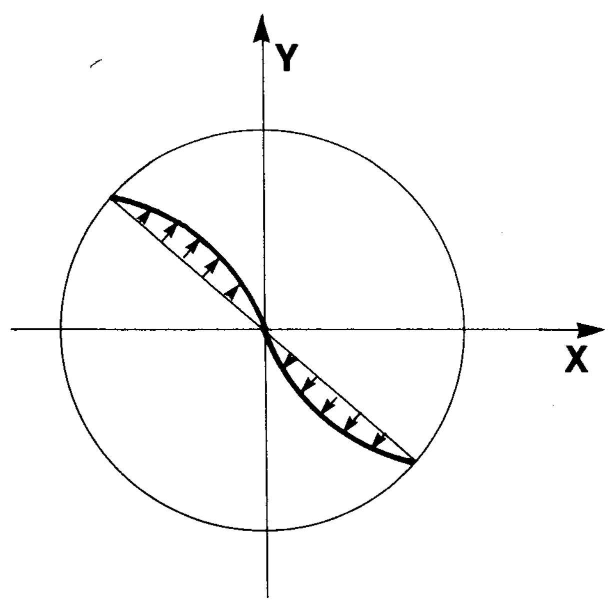

FIG. 8. The structure of an f-plane eddy with a parabolic velocity distribution [$\bar{v}_\theta = 2\,\mathrm{Ro} f_0 r(rr_0^{-1} - 1)$].

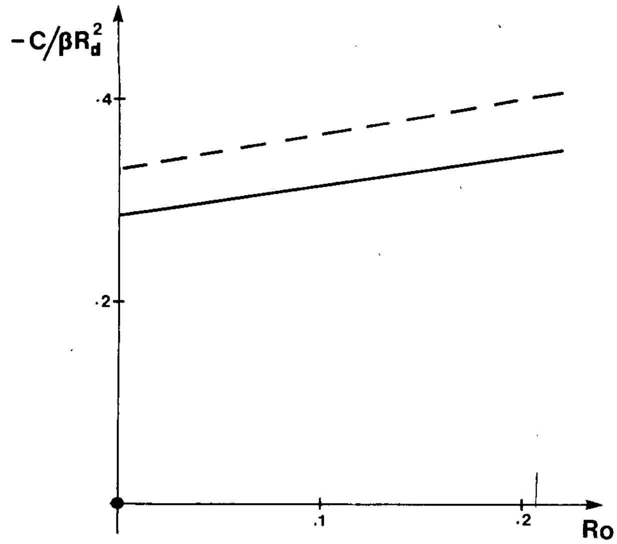

FIG. 9. A comparison between the predicted translation speed for two eddies with different structures. The solid line shows the translation of an eddy whose f-plane structure corresponds to a parabolic velocity distribution (5.4) and the dashed line shows the translation corresponding to a linear velocity distribution (4.5). Note that the two eddies have the same maximum depth $\hat{h}$ and the same Rossby number (Ro).

Note that, in contrast to the previously considered eddies with a linear velocity distribution, this eddy is not subject to a local instability around its edge (i.e., Ri > ¼ everywhere).

The translation speed can now be calculated by using Eqs. (3.6), (5.1), (5.2) and (5.3). For the Loop Current eddies, the Rossby number is of O(0.1) (see e.g., Elliott, 1979); while it is not entirely negligible, it is not very large and, therefore, one may neglect terms of O(Ro^2) in calculating the translation speed. Under these conditions, Eqs. (3.6), (5.1), (5.2) and (5.3) give

$$C = (0.285 + 0.306\,\mathrm{Ro})\beta R_d^2 + O\left(\frac{\beta^2 r_0 R_d^2}{f_0}\right) + O(\mathrm{Ro}^2 \beta R_d^2). \quad (5.4)$$

In terms of the ratio between the deformation radius and the radius of the eddy, (5.4) is

$$C \approx [0.285 + 0.918(R_d/r_0)^2]\beta R_d^2.$$

Relation (5.4) predicts translation speeds that are fairly close to those predicted for the eddies with linear velocity distribution. As can be seen from Fig. 9 the difference between the speed predicted by (5.4) and those predicted by (4.9), for two eddies having identical maximum depth and identical Ro, is less than 15% suggesting that the translation speed is not very sensitive to the velocity distribution within the eddy. This property is not very surprising because according to (3.5) the translation speed depends on the average value of the transport function which is sensitive to the amount of fluid circulating within the eddy, but is not very sensitive to its distribution.

Following Hulburt and Thompson (1980, hereafter referred to as HT) who investigated the formation and translation of Loop Current eddies numerically, we choose the following values as being typical for the Gulf of Mexico:

$$\left.\begin{array}{l}\beta \approx 2 \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}; \quad g' = 3 \times 10^{-2}\,\mathrm{m\,s}^{-1} \\ f_0 \approx 5 \times 10^{-5}\,\mathrm{s}^{-1}\end{array}\right\}. \quad (5.5)$$

For the purpose of comparing our results to those of HT we shall consider 28 of their "reduced gravity" numerical experiments. These experiments correspond to the 35 experiments discussed by HT (pp. 1618–1621), excluding seven experiments which are not relevant to our present analysis. Experiments RG3-RG7 cannot be included because they correspond to various values of β and f_0, and experiments RG18 and RG43 were excluded because their eddy amplitude is rather small compared to the upper layer depth.

Averaged over the 28 experiments, the diameter of the eddy (defined on the basis of distance to the region of maximum speed) and the maximum velocity are ~356 km and ~0.71 m s^{-1}, respectively. Together with (5.5) and (5.2), these values give an average Ro ≈ 0.08, whereas the values given by Elliott (1979) suggest that Ro ≈ 0.12. As a compromise we shall take 0.10 as being the average

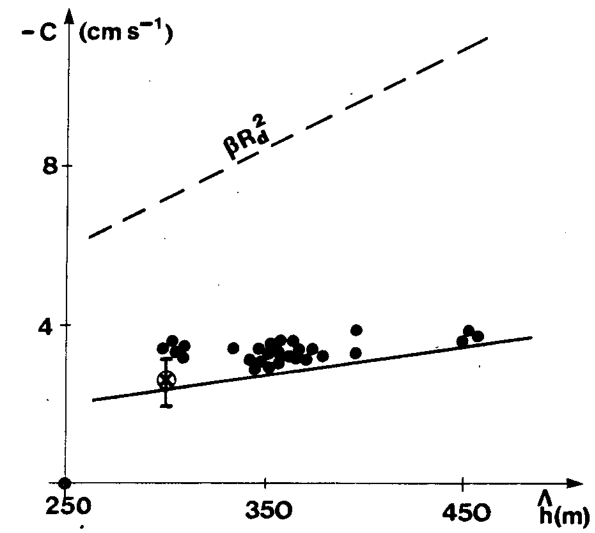

FIG. 10. The predicted β-induced translation of Loop Current eddies as a function of the maximum depth (solid line). Solid dots represent the numerical experiments of Hulburt and Thompson (1980). The circled cross corresponds to the average observed translation speed and its standard deviation as calculated by Elliott (1979) from several field observations. The Rossby wave speed (βR_d^2) is represented by the dashed line.

Rossby number for Loop Current eddies. For this Rossby number, the translation speed is found from (5.4) to be

$$C \approx 0.32\beta R_d^2. \qquad (5.6)$$

Before proceeding and discussing (5.6) and its implications, it should be pointed out that, in general, the Rossby number is a function of $\hat{h}$ so that the term in the brackets in (5.4) cannot be replaced by a constant. However, for the problem under consideration the influence of the Rossby number on the translation speed is relatively small and the dependence of Ro on $\hat{h}$ is weak so that replacement of the above term by a constant is allowed. The numerical experiments of HT indicate that replacement of the term in the bracket in (5.4) by a constant (0.32) does not introduce errors larger than 3%.

The predicted translation speed for the Loop Current eddies (5.6) is shown in Fig. 10 which also includes the translation found by HT in their numerical experiments, the speed of a simple Rossby wave (βR_d^2), and the average translation speed determined by Elliott (1979) from field observations. The latter has been taken to be associated with a maximum depth ($\hat{h}$) of ~300 m because the data presented by Elliott (1979) suggest a maximum depth which is between 250 and 350 m. It will become clear soon that the results are not very sensitive to this choice; the relationship between the observed speed and our model prediction would have only slightly changed if we had chosen the maximum depth to be 250 or 350 m.

Several comments should be made with regard to Fig. 10. First, we see that the westward translation is not properly described by βR_d^2; the numerical experiments of HT, the observed translation in the field and the present model all show that the translation is about one-third this speed. This is supported by the fact that the discrepancy between our model prediction and the numerical experiment of HT is <20% which is much better than the discrepancy between the numerical experiments and βR_d^2. Second, it appears that although our model is highly simplified, it predicts the actual translation speed as well as the numerical model does. Third, it should be pointed out that the discrepancy between the model prediction and the numerical experiments of HT probably results from various processes which are included in the numerical model but are absent from our model. For instance, as can be seen from Fig. 3 the eddy size is not very small compared to the size of the basin so interaction with the boundaries may affect the translation. In addition, there is an important difference between the HT's model and the present analysis; the numerical model of HT cannot take into account an interface which strikes the free surface so that their eddies reach a finite depth at the outer edge. And finally, friction is included in their model. These processes may well account for the differences in the predicted translation speed.

6. Summary

Prior to listing our conclusions, it is appropriate to stress again the main limitations and weaknesses of our proposed method for calculating the translation speed. Our model has been developed under the assumptions that: (i) the density of the eddy can be taken to be uniform, (ii) the depth vanishes along the outer edge, (iii) the motion is mainly frictionless and nondiffusive, (iv) the eddy is free and isolated, and (v) the eddy translates steadily without changing its shape with time. These assumptions are all supported by scaling arguments but there is no doubt that, in reality, under some circumstances they will not be valid. However, we saw that our proposed method predicts the correct translation speed for Loop Current eddies and that it enables one to examine the basic mechanics associated with isolated eddies.

The results of the study can be summarized as follows:

1) As the momentum of an anticyclonic baroclinic eddy whose depth vanishes along the outer edge translates westward, its entire mass anomaly is carried along with it. This conclusion is valid for both linear and nonlinear eddies.

2) The fully nonlinear β-induced translation of these lenslike eddies can be easily calculated using the relationship

$$C \approx \frac{-2\pi\beta}{f_0 Q}\int_0^{r_0}\int_{r_0}^{r} \bar{v}_\theta(r)\bar{h}(r)drrdr,$$

where the swirl speed $\bar{v}_\theta(r)$, the depth ($\bar{h}$) and the radius r_0 correspond to the structure that the eddy would have in the absence of β. Q is the volume of the eddy.

3) Nonlinear lens-like eddies translate considerably faster than linear lens-like eddies. However, the translation speed is always considerably smaller than that of a simple Rossby wave (βR_d^2).

4) For a parabolically distributed swirl velocity the translation speed is

$$C \approx -(0.285 + 0.306 \text{ Ro } \beta R_d^2,$$

where Ro[$\approx$O(0.1)] is defined on the basis of the maximum speed and its distance from the center. The proposed method for calculating the nonlinear β-induced translation 2) was applied to simply structured vortices with an approximate linear velocity distribution, and to more complicated eddies representing the Loop Current eddies observed in the Gulf of Mexico. For the Gulf of Mexico eddies, the predicted speed is about $0.32\beta R_d^2$ which compares very well with both numerical experiments and field observations.

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