Feb 1988

Geophys. Astrophys. Fluid Dynamics, Vol. 40, pp. 165–193
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Outflows Dynamics

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(*Received 3 March 1987*)

An analytical method for computing the classical problem of a light (or heavy) water source feeding the ocean near the coast is proposed. The nonlinear model includes two active layers; the width of the basin in which the outflow spreads is taken to be finite. Friction is neglected but the motions near the source are not constrained to be quasi-geostrophic. The velocity and pressure fields are computed by taking into account the flow forces to the left and right (looking off-shore) of the source, without assuming a hydrostatic pressure along the front and tail. This procedure leads to a set of five algebraic equations (with five unknowns) which can be solved analytically.

It is found that outflows (with a uniform potential vorticity), spreading in an infinitely deep ocean at rest can *never* reach a steady state. Furthermore, even if the natural tendency of the outflow to deflect to the right is aided by a current flowing from left to right, a steady state can still not be reached. Steady spreading states are possible only when the ocean is of *finite* depth and there is a long-shore current. The behavior of such steady outflows is sensitive to the distribution of potential vorticity.

When the outflow's potential vorticity is zero, all steady state solutions correspond to a deflection to the right, regardless of the ambient flow direction. An outflow with a *finite* potential vorticity behaves in quite a different way. When the outflow's potential vorticity is identical to that of the ambient fluid there are steady solutions corresponding to deflections to either the right or left. An unusual aspect of the outflow is that, for some range of parameters, *multiple equilibria* exist. Namely, the outflow deflects to either direction for the *same* ambient current and outflow's discharge. This implies that the outflow may flip from one position to the other.

Possible application of this theory to various oceanic situations is mentioned.

KEY WORDS: Outflow, longshore current, potential vorticity.

1. INTRODUCTION

When an inviscid light fluid is steadily released along a coast, a wedge-like current is formed due to the balance between the pressure gradient and the Coriolis force. Such a situation is created when a river is emptying into the ocean or a marginal sea releases its anomalous water. The current which is set up involves a flow along the coast on one side, and a front that intersects the coast on the other (Figure 1). The intersection of the front with the wall (i.e. the outflow's "tail") represents a region where the flow is nonhydrostatic because of the high ratio between the depth scale and the horizontal length scale. Our aim in this paper is to compute the current speed and width as a function of the source mass flux. As described below, this classical problem and variations of it have been addressed by many investigators during the last few decades.

It is expected that the width (and speed) will correspond to the deformation radius (and the internal gravity wave speed) based on the near wall depth which must be determined as a part of the problem. The general computations are quite difficult because of the inherent nonlinearity, the nonhydrostatic motions near the tail and the difficulty in finding steady solutions. The nonlinearity results from the fact that both the Rossby number and the depth variations are of order unity, and the nonhydrostatic motions are a consequence of the conditions in the vicinity of the outflow's tail. The absence of general steady solutions stems from the difficulty in balancing the along-shore flow-force with a form-drag (exerted on the outflow by the ambient fluid). We shall see that, despite these aspects, it is possible to isolate some steady solutions. To derive these solutions we shall balance the flow-force behind and ahead of the tail without assuming that the pressure is hydrostatic in between.

a) *Previous investigations* The first studies of river spreadings in the ocean are those of Takano (1954, 1955). His basin occupied half of the plane and the motions were frictionally dominated. Consequently, the river plume deflects to the right in the northern hemisphere. Paul and Lick (1974) studied numerically the river discharge into a lake with a pre-existing circulation, and Beardsley and Hart (1978) have looked at the way that estuarine flow interacts with the continental shelf. The reader is also referred to a class of studies which examine the circulation induced by the diffusion and advection of a fresh water source on the continental shelf (Csanady, 1985; Hendershott and Malanotte-Rizzoli, 1976; Shaw, 1982; Shaw and Csanady, 1983). Other investigations related to the problem posed

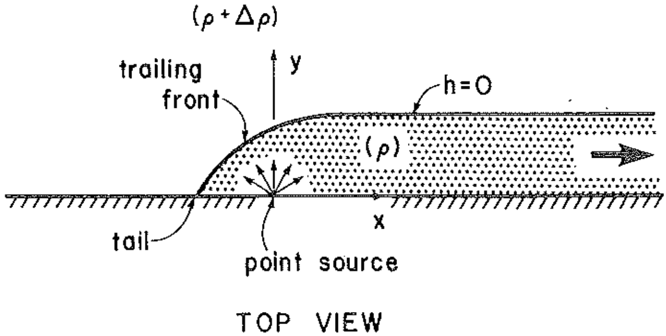

Figure 1 A sketch of the expected inviscid flow due to a fresh water source; the ocean is infinitely broad and infinitely deep. It is shown in the text that such an outflow can never have a steady solution because the forward flow-force associated with the outflow has no opposing force to balance it. Even if a uniform long-shore flow is added to the field, a steady solution is still impossible. However, when the width and depth of the ocean are finite (Figure 2) then there are steady solutions to the problem because the forward flow-force associated with the outflow can be balanced by a form-drag exerted (on the outflow) by the ambient fluid.

above are those of Nof (1978a, b), Nof (1981) and Nof (1987a) who considered outflows spreading in a channel and in a wedge-like basin. The study of Garvine (1987) and the transient outflow analysis of Whitehead and Miller (1979) also have some bearing on the process in question.

From a dynamical point of view, the intrusion studies of Stern (1980), Stern *et al.* (1982), Griffiths and Hopfinger (1983), Kubokawa and Hanawa (1984a, b), Griffiths (1986) and Nof (1987b) are also related to the problem at hand. These studies examine the behavior of the initial intrusion resulting from a fresh water source which is suddenly "turned on" by, say, the breaking of a dam. Namely, they focus on the intrusion which propagates along the coast immediately after the fresh water is released. For our present problem it is assumed that the intrusion head is located far away from the source (i.e. at $x \rightarrow \infty$) so that it has no influence on the vicinity of the source.

Despite the obvious differences (such as the lack of propagation in the spreading case, and the fact that the near wall depth is given in the intrusion case but is unknown in the spreading case), there is an important similarity between the intrusion and the spreading

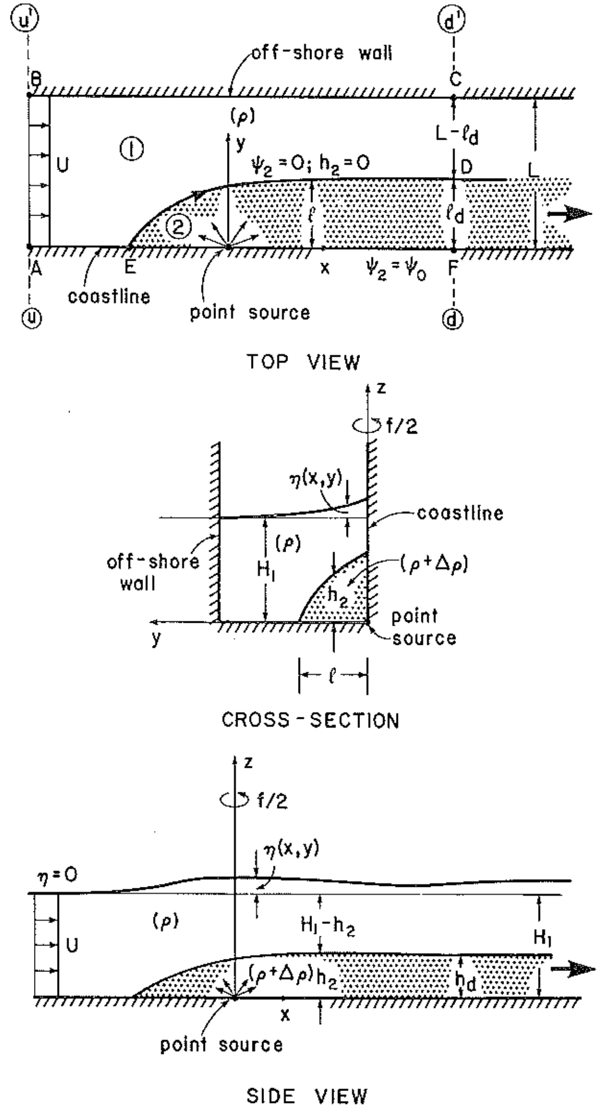

Figure 2 Schematic diagram of the model under study. Near the outflow's "tail" (point E) the flow is nonhydrostatic. Cross-sections "d′–d" and "u′–u" are associated with the downstream and upstream field; they are located several deformation radii away from the source. Note that the terms "downstream" and "upstream" are used with reference to the *outflow* and not the ambient fluid.

outflow. It turns out that, the outflow's *tail* (i.e. the point where the outflow's width vanishes) has a very similar structure to the intrusion's *nose* and the balance of flow-forces is also very similar. Because of these similarities, the techniques that have been adopted here are very similar to those discussed by Nof (1987b) for the dam-breaking problem. There is some (but limited) over-lapping between the two articles because an attempt has been made to make the present paper self-contained.

b) *Methods* As mentioned, the approach that will be taken in this paper is to equate the flow-force on the left (looking off-shore) and right side of the source. This provides a direct connection between the downstream and upstream fields and, together with conservation of energy and potential vorticity, enables one to compute the desired speed and width without assuming that the pressure near the tail is hydrostatic. Instead of considering the special case of an infinitely deep and infinitely broad ocean (Figure 1), we shall take the ocean (Figures 2 and 3) to be of finite width (L) and finite depth (H_1). The $H_1 \rightarrow \infty$ and $L \rightarrow \infty$ case will then be obtained by taking the appropriate limits of the more general case.

Our mathematical treatment involves integration of the momentum equations across the channel in which the outflow is spreading.

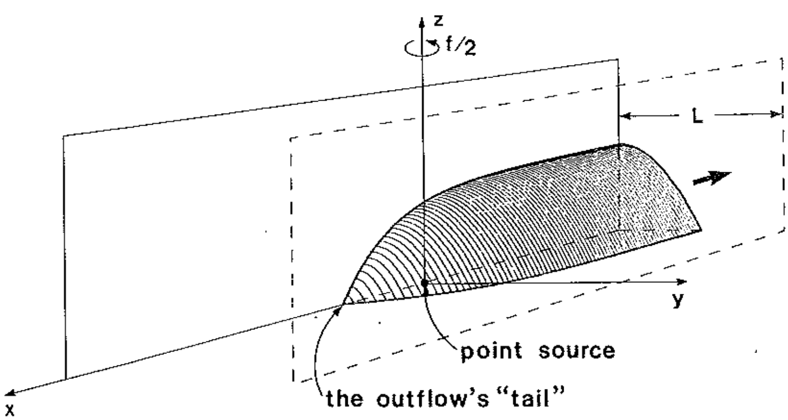

Figure 3 Three-dimensional view of an outflow deflecting to the left (looking off-shore). The solid rectangular corresponds to the coast and the dashed to the off-shore wall.

The potential vorticity equations for the outflow and the ambient fluid are then solved and the procedure provides a set of complicated algebraic equations which can be solved analytically. Because of the complexity of the associated integrals, the algebra was done using the so-called "Macsyma" program which enables one to obtain algebraic expressions using a computer (see Symbolics, 1985). This program has been extensively tested and assures that there are no algebraic errors. After the solution for a finite depth (H_1) and finite width (L) ocean is obtained, the limit of the solution when $H_1 \to \infty$ and $L \to \infty$ is examined and it is then shown that in this case steady solutions are impossible.

This paper is organized as follows. In Section 2 the formulation of the problem is presented; the governing equations are given in Section 3, and the relationships between the field behind and ahead of the tail are considered in Section 4. The solution is presented in Sections 5 and 6; Section 7 gives the analysis and a discussion of the problem, and the results are summarized in Section 8.

2. FORMULATION

As an idealized formulation of the problem consider again the situation shown in Figure 2. The outflow is spreading in an oceanic channel with a finite depth (H_1); for simplicity, it will be assumed that the spreading fluid has a uniform potential vorticity f/H_2. We shall consider heavy outflows spreading along the bottom of the oceanic channel. Recall, however, that light outflows spreading on the top are mathematically identical. The x and y axes are directed along and across the channel and the system rotates uniformly at $f/2$ about the vertical axis (z). The outflow's width and depth are denoted by l and h_2 (respectively) and the "tail" of the outflow is defined by $l = h_2 = 0$.

In a similar fashion to intrusions of light water along a coast [see, for example, Stern (1980), Stern *et al.* (1982), Griffiths (1986), Nof (1987b)] there is a stagnation point at the tail because a cross-section in the xz-plane indicates the presence of a discontinuity in the slope of the streamline associated with the surface in the lee of the outflow. The existence of such a stagnation point was used by Stern *et al.* (1982); it is also discussed by Griffiths (1986). The reader might be interested in knowing that for nonrotating intrusions Von Karman (1940) has elegantly shown that the discontinuity is associated with an intersection angle of 60°. As mentioned, the flow in the vicinity of the tail is nonhydrostatic because the horizontal scale is not necessarily larger than the vertical [see Griffiths (1986)].

3. GOVERNING EQUATIONS FOR THE UPSTREAM AND DOWNSTREAM REGIONS

We shall now determine the equations governing the flow several deformation radii away from the trailing edge (sections "u" and "d", Figure 2). In the lee of the outflow tail, in region "u", the speed is, by definition, identical to the advecting flow U, namely,

$$v_{u1}=0; \quad u_{u1}=U; \quad \eta_u=-fUy/g.$$

Here, u and v are the horizontal velocity components in the x and y direction, η is the free surface vertical displacement and h is the total depth. The subscript 1 indicates association with the ambient fluid and the subscript "u" indicates that the variable is associated with the "upstream" field. We shall later use a subscript "d" to indicate association with the "downstream" field. Note that the upstream and downstream fields are defined on the basis of their relationship to the *outflow* and not the ambient flow.

Within the outflow, in region "d", the flow is geostrophic (due to the presence of the wall) and, as mentioned earlier, has uniform potential vorticity so that,

$$-\partial u_{d2}/\partial y+f=h_{d2}f/H_2. \tag{3.1}$$

Here, the subscript 2 denotes association with fluid 2 (i.e. the outflow). The pressure in the outflow is $p_2=\rho g(\eta+H_1-z)+g\Delta\rho(h_2-z)$ so that the momentum balance for the outflow is given by

$$fu_{d2}=-g(\partial\eta_d/\partial y)-g'(\partial h_{d2}/\partial y), \tag{3.2}$$

where η is the free surface displacement and g' is the "reduced gravity," $g\Delta\rho/\rho$. For rigid lid motions ($\eta_d \ll h_2$), the downstream upper layer is governed by,

$$[-(\partial u_{d1}/\partial y)+f]/(H_1-h_{d2})=f/H_1, \tag{3.3}$$

$$fu_{d1}=-g\,\partial\eta_d/\partial y. \tag{3.4}$$

Here, we have taken into account that the upper layer fluid has originated behind the outflow where the potential vorticity is f/H_1.

The boundary conditions for the outflow (fluid 2) are,

$$h_{d2}=0;\quad y=l_d \tag{3.5a}$$

$$h_{d2}=\hat{h}_{d2};\quad y=0. \tag{3.5b}$$

For the upper fluid there are no independent boundary conditions; the only boundary conditions that it should satisfy are those associated with the connection of the upstream and downstream fields which will be discussed in the next section.

4. THE CONNECTION BETWEEN THE UPSTREAM AND DOWNSTREAM FIELDS

As mentioned earlier, we shall obtain the desired solution to the problem without solving for the complicated nonhydrostatic (three-dimensional) field near the trailing edge. We will accomplish this by using the following connection principles.

a) *The flow force* This is obtained by integrating the x momentum equation across the channel away from the outflow's tail. Specifically, the integration is done where the flow is *hydrostatic* but we do not assume yet that it is one dimensional (i.e. independent of x as is the case in cross-section d–d'). For the outflow we have,

$$\iint_s\left(u_2\frac{\partial u_2}{\partial x}+v_2\frac{\partial u_2}{\partial y}\right)dy\,dz-f\iint_s v_2\,dy\,dz$$

$$=-\iint g\frac{\partial\eta}{\partial x}dy\,dz-\iint g'\frac{\partial h_2}{\partial x}dy\,dz, \tag{4.1}$$

where s is the cross-section. Since u_2 and v_2 are hydrostatic and independent of z, (4.1) can also be written in the form,

$$\int_0^{l(x)} [\partial(h_2u_2^2)/\partial x + \partial(h_2u_2v_2)/\partial y]\, dy - \int_0^{l(x)} fv_2h_2\, dy + \int_0^{l(x)} gh_2(\partial\eta/\partial x)\, dy + \tfrac{1}{2}g' \int_0^{l(x)} (\partial h_2^2/\partial x)\, dy = 0, \tag{4.2}$$

where $l(x)$ is the outflow width and the continuity equation has been used to simplify the expression for the nonlinear terms. By defining the stream function ψ_2,

$$\partial\psi_2/\partial y = -u_2h_2, \qquad \partial\psi_2/\partial x = v_2h_2,$$

(4.2) can be written as

$$h_2u_2v_2\Big|_0^{l(x)} + \int_0^{l(x)} \left[\frac{\partial}{\partial x}(h_2u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2)\right] dy + \int_0^{l(x)} gh_2\frac{\partial\eta}{\partial x}\, dy = 0,$$

which, since $h_2 = 0$ along $y = l(x)$ and $v_2 = 0$ along $y = 0$, reduces to

$$\int_0^{l(x)} [(\partial/\partial x)(h_2u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2)]\, dy + \int_0^{l(x)} gh_2(\partial\eta/\partial x)\, dy = 0.$$

This equation can be further simplified by using the Leibnitz rule for the differentiation of an integral which gives,

$$(\partial/\partial x)\left\{\int_0^{l(x)} (h_2u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2)\, dy\right\} - [h_2u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2]_{y=l(x)}(\partial l/\partial x) + \int_0^{l(x)} gh_2(\partial\eta/\partial x)\, dy = 0. \tag{4.3}$$

Note that the term in the square brackets is to be evaluated along the edge ($h_2 = 0$). By defining ψ_2 to be zero along the edge, (4.3) takes the form,

$$(\partial/\partial x)\left\{\int_0^{l(x)} (h_2u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2)\, dy\right\} + \int_0^{l(x)} gh_2(\partial\eta/\partial x)\, dy = 0. \tag{4.4}$$

By applying the same principles to the ambient fluid [along the same cross-section to which (4.1) is applied] one finds

$$(\partial/\partial x)\left\{\int_0^L (h_1 u_1^2 - f\psi_1 + gH_1\eta)\,dy\right\} - \int_0^{l(x)} gh_2(\partial\eta/\partial x)\,dy = 0, \quad (4.5)$$

where we have taken into account that the oceanic channel has a constant width (L). By adding (4.4) and (4.5) we have,

$$\underbrace{(\partial/\partial x)\left\{\int_0^L (h_1 u_1^2 - f\psi_1 + gH\eta)\,dy + \int_0^{l(x)} (h_2 u_2^2 - f\psi_2 + \tfrac{1}{2}g'h_2^2)\,dy\right\}}_{\mathrm{I}} = 0, \quad (4.6)$$

which states that if the momentum is conserved then the *flow-force* (I) does not vary from one cross-section to another. Recall that (4.6) was derived for regions where the flow is hydrostatic. However, since the term in the brackets represents the flow-force, it is clear from a physical point of view that it can be applied to cross-sections "u" and "d" even though the flow in between the cross-sections is *not* hydrostatic.

In view of this, application of (4.6) to the downstream ($l = l_d$) and upstream ($l = 0$) sections gives,

$$\int_0^L [(H_1 - h_{d2})u_{d1}^2 - f\psi_{d1} + gH_1\eta_d]\,dy + \int_0^{l_d} (h_{d2}u_{d2}^2 - f\psi_{d2} + \tfrac{1}{2}g'h_{d2}^2)\,dy$$
$$= \int_0^L (Hu_{u1}^2 - f\psi_{u1} + gH_1\eta_u)\,dy, \quad (4.7)$$

which states that the forward flow-force associated with the outflow is balanced by a form-drag exerted (on the outflow) by the ambient fluid. Note that in applying (4.6) to the downstream and upstream fields, it has been assumed that the point source does not directly contribute to the integrated momentum in the x direction. We shall see later at the end of Section 7 that this plausible assumption implies that $\int huv\,dx = 0$ across the source (see Figure 4).

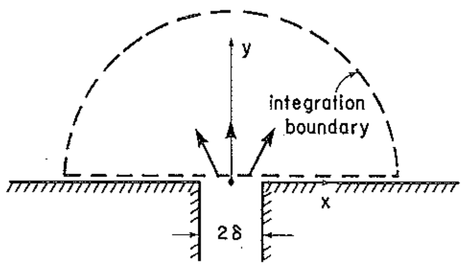

Figure 4 A "close-up" of the source when the feeding channel width is not zero ($\delta \neq 0$). It is shown in the end of Section 7 that the contribution of the source to the integrated x momentum equation (through the term $\int_{-\delta}^{\delta} huv\,dx$) vanishes. The dashed line corresponds to the integration discussed in Section 7.

b) *Continuity* For the outflow, the continuity equation gives

$$\int_0^{l_d} h_{d2} u_{d2}\, dy = \psi_0, \tag{4.8}$$

where ψ_0 is the source's discharge. Similarly, we have for the ambient fluid,

$$\int_0^L H_1 u_{u1}\, dy = \int_0^L (H_1 - h_{d2}) u_{d1}\, dy. \tag{4.9}$$

c) *Energy* Since we are seeking solutions for energy conserving flows, we may apply the Bernoulli integral along all streamlines. For hydrostatic motions, the Bernoulli invariant is

$$\tfrac{1}{2}(u^2 + v^2) + p/\rho + gz = B(\psi), \tag{4.10}$$

where p is the pressure and z is the height. Application of (4.10) to the ambient fluid along the channel walls ($y = L, 0$) gives

$$\tfrac{1}{2}u_{u1}^2 + g\eta_u = \tfrac{1}{2}u_{d1}^2 + g\eta_d; \quad y = L, \tag{4.11}$$

$$\tfrac{1}{2}u_{u1}^2 = \tfrac{1}{2}u_{d1}^2 + g\eta_d; \quad y = 0, \tag{4.12}$$

where, as mentioned before, the regions where we have applied (4.10) are hydrostatic because they are far from the trailing edge. Note that a similar procedure was used by Benjamin (1968).

d) *Stagnation point at the tail* In addition to the constraints mentioned above, there is a constraint resulting from the fact that the tail is a double stagnation point. Specifically, as the relatively light ambient fluid "climbs" on top of the outflow at the tail it senses a discontinuity in the "bottom" slope, implying that the speed vanishes there. A similar situation exists at the nose of an intrusion along the coast [see, for example, Stern *et al.* (1982), Griffiths (1986), Nof (1987b)]. Because of the wall, the heavy outflow fluid must also feel such a discontinuity and, therefore, it must also stagnate at the tail. These properties can be combined with the Bernoulli integral to obtain additional relationships between the variables that we seek. However, the flow in the vicinity of the tail is nonhydrostatic (because of the discontinuous "bottom" slope) so that we can only use the most general form of the Bernoulli invariant which is not restricted to hydrostatic flows. This general form is

$$\tfrac{1}{2}(u^2+v^2+w^2)+p/\rho+gz=B_i, \tag{4.14}$$

where B_i is a constant which varies from one streamline to another. Note that because of the three dimensions, B_i cannot be written in terms of a streamfunction.

Application of (4.14) to the *ambient* fluid along the streamline connecting the stagnation point and point F (Figure 2) gives

$$p_E/\rho=\tfrac{1}{2}u_{d1}^2+g(\eta_d+H_1-\hat{h}_{d2})+g\hat{h}_{d2}, \tag{4.15}$$

where p_E is the nonhydrostatic pressure at E. Similarly, application of (4.14) to the *outflow* fluid along ED gives,

$$p_E/(\rho+\Delta\rho)=\tfrac{1}{2}u_{d2}^2+g(\eta_d+H_1)\rho/(\rho+\Delta\rho). \tag{4.16}$$

Multiplication of (4.16) by $(\rho+\Delta\rho)/\rho$ and subtracting the resulting equation from (4.15) gives

$$[\tfrac{1}{2}u_{d1}^2+g\eta_d]_{y=0}=[\tfrac{1}{2}u_{d2}^2+g\eta_d]_{l_d}, \tag{4.17}$$

where, in accordance with the Boussinesq approximation, we have neglected terms of $O(u_{d2}^2\Delta\rho/\rho)$.

We shall see in the next sections that the set of constraints and boundary conditions given in (a)–(d) is sufficient for solving the problem. Although not *a priori* obvious, some of the constraints associated with the conservation of energy are automatically satisfied, as we shall shortly see.

5. GENERAL SOLUTION

a) *The upstream and downstream fields* As already mentioned, the solution upstream is

$$u_{u1}=U, \quad v_{u1}=0, \quad h_{u1}=H_1+\eta_u, \quad \psi_{u1}=-UH_1y, \quad \eta_u=-fUy/g, \tag{5.1}$$

where we defined ψ_{u1} to be zero along the coast ($y=0$) and we have taken into account that $\eta \ll H$.

For the downstream region we divide the cross-section associated with the ambient fluid into two parts. The ambient fluid that is directly above the outflow (section DF, Figure 2) will be referred to as the *inner* fluid whereas the ambient fluid that is in direct contact with the bottom (region CD, Figure 2) will be referred to as the *outer* fluid.

In the outer region there is only one fluid so that with the aid of the rigid lid approximation, (3.3) and (3.4) give

$$u_{d1}^{(0)}=\text{constant}, \qquad \eta_d^{(0)}=\eta_{dD}-fg^{-1}u_{d1}^{(0)}(y-l_d), \tag{5.2}$$

where the superscript zero (^{0}) indicates that the variable in question is association with the "outer solution" (region CD), and the subscript "D" denotes association with point D ($y=l_d$). Note that (5.2) assures the continuity of η (i.e. pressure) across D. For the inner region, the solution is also straightforward but is slightly more complicated than (5.2) because there are now two active fluids whose motions are coupled. It is easy to form a single equation for h_{d2},

$$\partial^2 h_{d2}/\partial y^2-(f^2/g')(H_1^{-1}+H_2^{-1})h_{d2}=-f^2/g',$$

which gives the solution

$$h_{d2}=-(g/g')(Ae^{-y/R_d}+Be^{y/R_d})+H_1H_2(H_1+H_2)^{-1}, \tag{5.3a}$$

where

$$R_d = [g'H_1H_2(H_1+H_2)^{-1}]^{1/2}/f.$$

The equation for the free surface displacement is

$$\partial^2\eta_d^{(i)}/\partial y^2 + (f^2/gH_1)\{H_1H_2(H_1+H_2)^{-1} - (g/g')[Ae^{-y/R_d} + Be^{y/R_d}]\} = 0$$

and the corresponding solutions are

$$\eta_d^{(i)} = \frac{-f^2H_2y^2}{2g(H_1+H_2)} + \frac{f^2R_d^2}{g'H_1}[Ae^{-y/R_d} + Be^{y/R_d}] - \frac{fCy}{g} - \frac{g'H_1H_2}{g(H_1+H_2)} - \frac{D}{g}, \qquad (5.3b)$$

$$u_{d1}^{(i)} = \frac{fH_2y}{(H_1+H_2)} + \frac{gfR_d}{g'H_1}[Ae^{-y/R_d} - Be^{y/R_d}] + C, \qquad (5.3c)$$

$$u_{d2} = \frac{gfR_d}{g'H_1}\left(1 - \frac{g'H_1}{f^2R_d^2}\right)[Ae^{-y/R_d} - Be^{y/R_d}] + \frac{fH_2y}{H_1+H_2} + C, \qquad (5.3d)$$

where C and D are constants of integration. Here, the superscript "i" indicates association with the inner part (region DF, Figure 2), A, B, C and D are constants to be determined from the boundary conditions.

The edge conditions

$$h_{d2} = 0 \quad \text{at} \quad y = l_d, \qquad h_{d2} = \hat{h}_{d2} \quad \text{at} \quad y = 0, \qquad (5.4)$$

[where the hat ($\hat{}$) denotes association with the coast $y=0$] can be used [with the aid of (5.3a)] to obtain the constants A and B in terms of l_d and $\hat{h}_{d2}$. One finds

$$A + B = (g'/g)[H_1H_2(H_1+H_2)^{-1} - \hat{h}_{d2}],$$
$$Ae^{-l_d/R_d} + Be^{l_d/R_d} = g'H_1H_2/g(H_1+H_2), \qquad (5.5)$$

which give

$$A = \frac{g' H_1 H_2 (e^{l_d/R_d} - 1)(H_1 + H_2)^{-1} - g' \hat{h}_{d2} e^{l_d/R_d}}{g(e^{l_d/R_d} - e^{-l_d/R_d})}, \tag{5.6a}$$

$$B = \frac{g' H_1 H_2 (1 - e^{-l_d/R_d})(H_1 + H_2)^{-1} + g' \hat{h}_{d2} e^{l_d/R_d}}{g(e^{l_d/R_d} - e^{l_d/R_d})}. \tag{5.6b}$$

Also, (5.4) and (5.3a) can be combined to give

$$\eta_{dD} = \frac{-f^2 H_2 l_d^2}{2g(H_1 + H_2)} - \frac{f C l_d}{g} - \frac{D}{g} + \frac{g' H_1 H_2}{g(H_1 + H_2)} \left(\frac{f^2 R_d^2}{g' H_1} - 1 \right). \tag{5.6c}$$

There is an additional condition which one should use and this is the continuity of the ambient fluid velocity at D. Namely, the outer and inner solutions should be matched at D and this gives

$$u_{d1}^{(0)} = (g f R_d (g' H_1)^{-1} (A e^{-l_d/R_d} - B e^{l_d/R_d}) + C + f H_2 l_d (H_1 + H_2)^{-1}. \tag{5.7}$$

b) *Connecting the upstream and downstream fields* The solutions (5.1)–(5.7) can now be substituted into the general connection equations derived in Section 4. The relevant flow-force equation is obtained by taking into account the condition of energy conservation along the off-shore wall (4.11). One can then derive the corresponding continuity equations for the ambient fluid (4.9) and the outflow (4.8), and the energy equations along the coast (4.12).

It will become clear later (once the solution is obtained) that the energy equation along the off-shore wall (4.11) is *automatically* satisfied so that the only remaining constraint is the one associated with the double stagnation (4.17) which can also be expressed in terms of the upstream and downstream solutions (5.1)–(5.7). With the above procedure, one finds a set of five (5) algebraic equations with five unknowns C, U, L, D and $\hat{h}_d$ so that it is possible to obtain the desired solution to the problem. Hereafter, these five independent equations will be referred to as the modified flow-force equation, the modified continuity equations for the ambient flow and the outflow, the modified energy equation for the coastline, and the modified stagnation equation.

The solution of the five modified equations is straightforward but tedious. The steps leading to the solution are briefly described in the next section; the reader who is only interested in the results may turn directly to Section 7. As mentioned in the Introduction, the "Macsyma" program has been used to derive the analytical solution. Because of the massive amount of algebra, it is doubtful that even the most patient mathematician could have derived the solution without such a program.

6. DETAILED SOLUTION

For convenience, we introduce the following nondimensional variables,

$$\left.\begin{array}{lll} C^*=C/(g'H_1)^{1/2}, & l_d^*=l_d/R_d, & L^*=L/R_d, \\ A^*=Ag/g'H_1, & B^*=Bg/g'H_1, & \bar{u}_{d1}^{(0)}=u_{d1}^{(0)}/(g'H_1)^{1/2}, \\ D^*=D/(g'H_1), & \eta_d^*=\eta_d g/g'H_1, & \psi_0^*=f\psi_0/g'H_1^2, \\ K=H_2/(H_1+H_1), & U^*=U/(g'H_1)^{1/2}, & \hat{h}_{d2}^*=\hat{h}_{d2}/H_1. \end{array}\right\} \quad (6.1)$$

To obtain the solution one proceeds as follows. First, we note that for mathematical convenience l_d^* is taken to be known so that our solution will consist of U^*, C^*, L^*, D^* and $\hat{h}_{d2}^*$ expressed in terms of l_d^*.

The solution for $\hat{h}_{d2}^*$ can be immediately obtained. Specifically, elimination of C^* between the modified continuity equation for the outflow and the modified stagnation constraint gives a cubic equation for $\hat{h}_{d2}^*$,

$$(\hat{h}_{d2}^*)^3+m_1(\hat{h}_{d2}^*)^2+m_2(\hat{h}_{d2}^*)+m_3=0, \quad (6.2)$$

where

$$m_1=\{K(1-2K)l_d^*+K(5K-1)e^{5l_d^*}+[K(2K+1)l_d^*-17K^2+5K]e^{4l_d^*}$$
$$+2K(l_d^*+13K-9)e^{3l_d^*}+2K(l_d^*-13K+9)e^{2l_d^*}$$
$$+K[(2K+1)l_d^*+17K-5]e^{l_d^*}+K(1-2K)l_d^*-K(5K-1)\}$$

$$/\{(e^{l_d^*}-1)(e^{2l_d^*}+1)(2Ke^{l_d^*}-Ke^{2l_d^*}-2e^{l_d^*}-K)\}, \tag{6.3a}$$

$$\begin{aligned}m_2=\{&-K[K^2[(l_d^*)^2-6l_d^*+8]-K-2\psi_0^*]e^{5l_d^*}\\&+[-K^3[(l_d^*)^2+10l_d^*-32]+K^2[4l_d^*-13]+2\psi_0^*(2-K)]e^{4l_d^*}\\&+[K^3(4l_d^*-56)+K^2(-4l_d^*+34)+4\psi_0^*(1-K)]e^{3l_d^*}\\&+[K^3(4l_d^*+56)+K^2(-4l_d^*-34)+4\psi_0^*(K-1)]e^{2l_d^*}\\&+[K^3[(l_d^*)^2-10l_d^*-32]+K^2[4l_d^*+13]+2K\psi_0^*-4\psi_0^*]e^{l_d^*}\\&+K^3[(l_d^*)^2+6l_d^*+8]-K^2-2K\psi_0^*\}\\&/\{(e^{l_d^*}-1)(e^{2l_d^*}+1)(2Ke^{l_d^*}-Ke^{2l_d^*}-2e^{l_d^*}-K)\},\end{aligned} \tag{6.3b}$$

and

$$\begin{aligned}m_3=\{&2K\psi_0^*(e^{2l_d^*}-1)^2([(K-1)l_d^*-2K+1]e^{l_d^*}+(K-1)l_d^*+2K-1)\\&+K^3(e^{l_d^*}-1)^3(l_d^*e^{l_d^*}-2e^{l_d^*}+l_d^*+2)\\&\times(Kl_d^*e^{l_d^*}-2Ke^{l_d^*}+e^{l_d^*}+Kl_d^*+2K-1)\}\\&/\{(e^{l_d^*}-1)(e^{2l_d^*}+1)(2Ke^{l_d^*}-Ke^{2l_d^*}-2e^{l_d^*}-K)\}.\end{aligned} \tag{6.3c}$$

The physically relevant roots of (6.2) are derived analytically subject to the condition that a physical root cannot lead to (i) an inflow (for any $y^*<l_d^*$), (ii) negative depth ($\hat{h}_{d2}$) or negative length (L^*), or (iii) complex variables.

Manipulation of the physically relevant roots of (6.2) with the modified continuity equations for the ambient fluid and outflow gives the solution for U^* and C^*,

$$U^*=F_2/L^*+F_3, \qquad C^*=-F_1, \tag{6.4, 5}$$

where

$$F_1=-\{(B^*)^2(K-1)e^{4l_d^*}+2B^*K(1-Kl_d^*)e^{3l_d^*}$$

$$+[K^3(l_d^*)^2+[-B^*(B^*+2)-A^*(A^*+2)]K-2\psi_0^*+(B^*)^2$$

$$+(A^*)^2]e^{2l_d^*}+2A^*K(Kl_d^*+1)e^{l_d^*}+(A^*)^2(K-1)\}$$

$$/\,K^{1/2}\{2B^*e^{3l_d^*}+2(-Kl_d^*-B^*+A^*)e^{2l_d^*}-2A^*e^{l_d^*}\}, \tag{6.6a}$$

$$F_2=-\tfrac{1}{2}e^{-2l_d^*}\{(B^*)^2K^{1/2}e^{4l_d^*}+2(B^*)K^{1/2}(1-Kl_d^*)e^{3l_d^*}$$

$$+[K^{3/2}(K-1)(l_d^*)^2-[B^*(B^*+2)+A^*(A^*+2)]K^{1/2}]e^{2l_d^*}$$

$$+2A^*K^{1/2}(Kl_d^*+1)e^{l_d^*}+(A^*)^2K^{1/2}\}$$

$$-F_1e^{-l_d^*}[B^*e^{2l_d^*}+[(1-K)l_d^*-B^*+A^*]e^{l_d^*}-A^*]-l_d^*F_3, \tag{6.6b}$$

$$F_3=K^{1/2}(A^*e^{-l_d^*}-B^*e^{l_d^*})+K^{3/2}l_d^*-F_1. \tag{6.6c}$$

Recall that A^* and B^* are given by the nondimensional version of (5.6). Note that, in contrast to $\hat{h}_d^*$ which is already given in terms of l_d^* alone, the solution that we have found so far for U^* and C^* is given in terms of both L^* and l_d^*. The relationship between L^* and l_d^* will be shortly provided by the modified flow-force equation.

We proceed now by computing D^* (also in terms of L^* and l_d^*) from the modified energy equation along the coast. One finds

$$D^*=-\tfrac{1}{2}(F_2/L^*+F_3)^2+(A^*+B^*)K-K+\tfrac{1}{2}[(A^*-B^*)K^{1/2}-F_1]^2. \tag{6.7}$$

By substituting the computed values of C^*, D^* and U^* into the modified flow-force equation, one finds

$$aL^*+b=0. \tag{6.8}$$

Here $a=a(l_d^*, \hat{h}_d^*)$, $b=b(l_d^*, \hat{h}_d^*)$ are given by

$$a=4B^{*3}(1-2K)e^{6l_d^*}+3B^{*2}[2(K^{1/2}(K-1)+2K^2)l_d^*$$

$$+K^{1/2}(1-4F_1)-3K]e^{5l_d^*}$$

$$+12B^*[-K^{5/2}l_d^{*2}+(K^{1/2}+F_1)Kl_d^*+A^*B^*(2K-1)]e^{4l_d^*}$$
$$+\{2[-2K^4+(3K^{1/2}+1)K^3-K^{5/2}]l_d^{*3}$$
$$+6[2F_1(K^{1/2}-1)K^2+F_1(1-K^{1/2})K]l_d^{*2}$$
$$+6[-K^3+2A^*(K^2+B^*(1-3K)K+KF_1-K^{3/2}(F_1+1))$$
$$+2B^*K(K+(F_1-1)K^{1/2}-F_1)+(A^{*2}+B^{*2})(K^2-K^{3/2})$$
$$+(F_3^2-F_1^2)K]l_d^*+3A^{*2}[4B^*(2K-1)-3K+(4F_1+1)K^{1/2}]$$
$$+3B^{*2}[3K+(4F_1-1)K^{1/2}]$$
$$+4B^{*3}(2K-1)+4A^3(1-2K)+12A^*B^{*2}(1-2K)+12F_2F_3K\}e^{3l_d^*}$$
$$+12A^*[K^{5/2}l_d^{*2}+(K^{1/2}-F_1)Kl_d^*+A^*B^*(1-2K)]e^{2l_d^*}$$
$$+A^{*2}[6(2K^2+K^{1/2}(K-1))l_d^*+3(3K-(4F_1+1)K^{1/2})]e^{l_d^*}$$
$$+4A^{*3}(2K-1), \qquad (6.9a)$$

and

$$b=6F_2^2e^{3l_d^*}. \qquad (6.9b)$$

With the derivation of L^* in terms of l_d^* the solution is now complete.

The expressions for the various variables are as follows: L^* is given by (6.8) with $\hat{h}_{d2}$ given by (6.2), U^* by (6.4), C^* by (6.5), and D^* by (6.7). Application of this solution to a zero potential vorticity outflow and a finite potential vorticity outflow is discussed in detail in the next section.

7. ANALYSIS

The solution is shown graphically for two cases, a zero potential vorticity outflow and a finite potential vorticity outflow. It will become clear from the following discussion that there are rather important differences between cases of different potential vorticity.

a) *A zero potential vorticity outflow* The solution for this case is shown in Figure 5a–c. A steady deflection to the left is impossible even if the ambient fluid is flowing from right to left (Figure 5c). Note, however, that this does not mean that an actual outflow will never deflect to the left. It merely implies that such outflows will exhibit some kind of time dependent motions such as oscillations or eddy shedding. Under such conditions, the numerical values of the parameters U^* and Q^* must lie *outside* the shaded areas shown in Figure 5c. For moderate and small fluxes [i.e. $Q^* \sim O(1)$ or $O(0.1)$], steady deflections to the right are established (Figure 5c). To determine the solution graphically, one begins with Figure 5a to determine l_d^* from Q^* and U^*. The remaining variables [i.e. the outflow near wall depth $(\hat{h}_{d2}^*)$, the speed near the coast, $u_{d2}^*(0)$, the speed along the edge, $u_{d2}^*(l_d^*)$, and the channel width, L^*] are determined from Figure 5b.

b) *A finite potential vorticity outflow* The solution for $K = \frac{1}{2}$ (i.e. outflow and ambient fluid with identical potential vorticities f/H_1) is

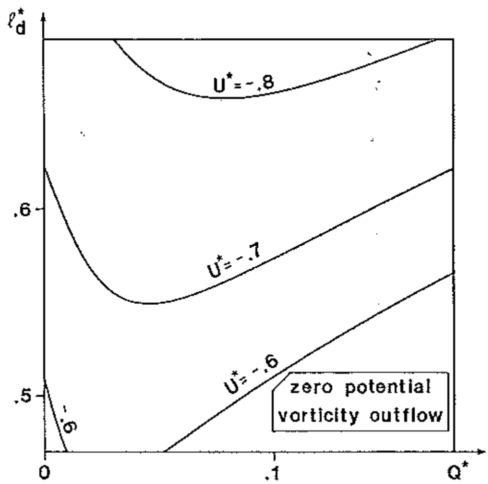

Figure 5a The solution for a zero potential vorticity outflow ($K = 1.0$). For each ambient speed (U^*) and each outflow's mass flux (Q^*), there is an outflow width (l_d^*). With this value of l_d^* one proceeds to Figure 5b to obtain the remaining variables. Note that $U^* < 0$ corresponds to an ambient flow from right to left. The outflow deflects to the right for the entire range of U^* and Q^*. The discharge Q^* is defined by $Q^* = |\psi_0^*| = \left|\int_0^{l_d^*} u_d^* h_d^* \, dy\right|$. Steady solutions corresponding to a deflection to the *left* are impossible (Figure 5c).

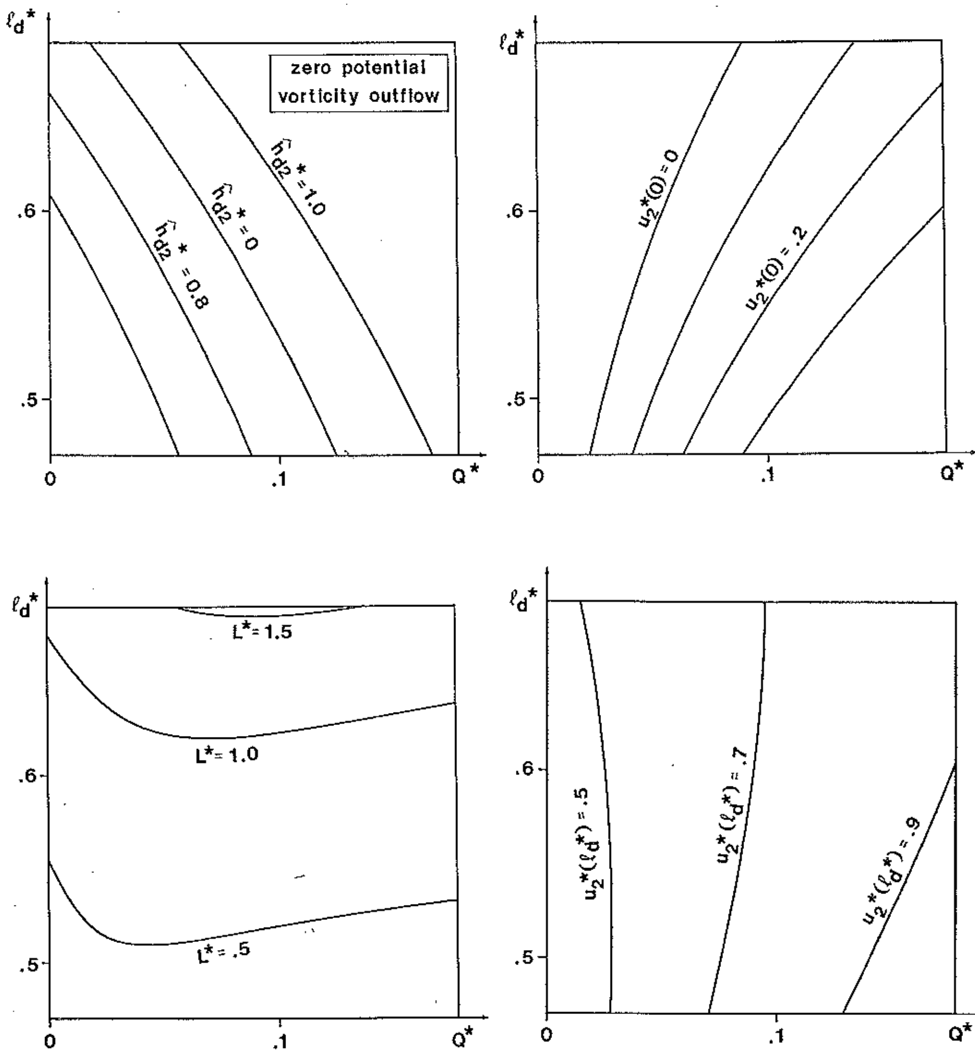

Figure 5b The dependence of the near wall outflow depth $\hat{h}_{d2}^*$ (upper left), the channel width L^* (lower left), the near wall speed $u_2^*(0)$ (upper right), and the edge speed $u_2^*(l_d^*)$ (lower right) on the outflow's discharge (Q^*) and width. Note that l_d^* is determined from Figure 5a and that the outflow deflects to the right. As mentioned in the text, for mathematical convenience, l_d^* is taken to be known whereas L^* is taken to be *unknown*.

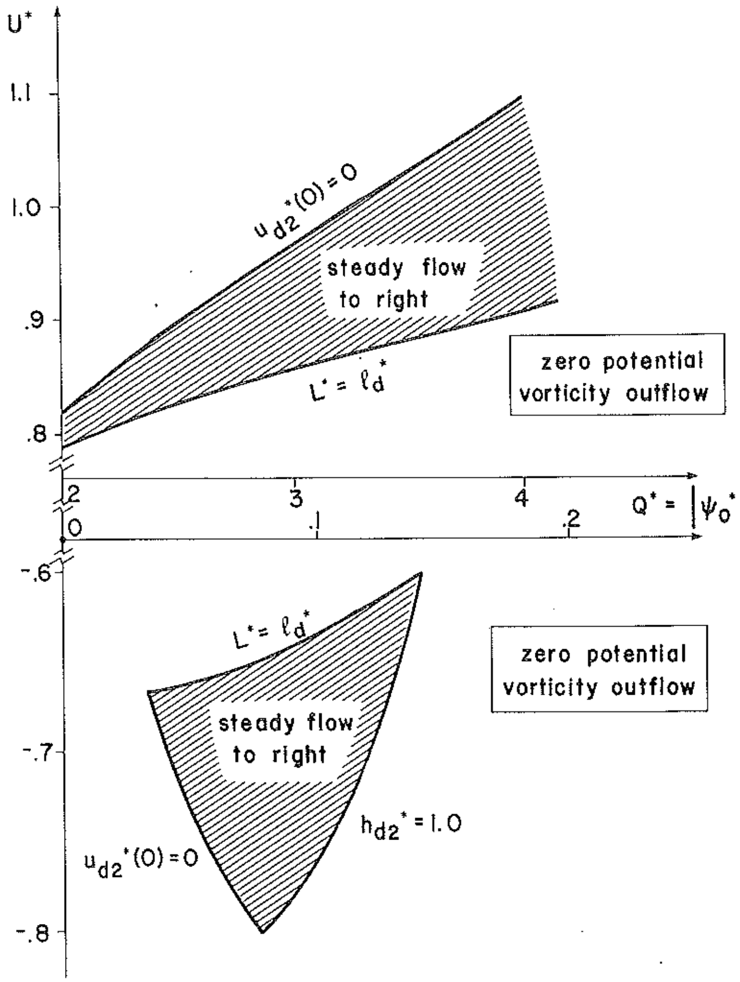

Figure 5c The various flows regimes as a function of the ambient fluid speed (U^*) and the outflow's discharge (Q^*). Shaded areas denote the existence of a steady solution; the outflow has zero potential vorticity. For simplicity only the regime corresponding to the lower panel is shown in Figures 5a, b. Positive and negative values of U^* correspond to an ambient fluid flowing from left to right and right to left, respectively. The nature of the unsteady flow (i.e. the unshaded areas) is unknown but one can speculate that it probably involves some waves on the outflow front.

shown in Figure 6. Note that the outflow deflects to the left when both the outflow's mass flux Q^* and the ambient current (flowing from right to left) are moderate (see Figure 6a, 6b). The deflection to the left results from the coastal current which overcomes the outflow's natural tendency to deflect to the right. While this behavior can be expected, the outflow behaves in a peculiar fashion when the discharge (Q^*) is ~0.1. Specifically, when the outflow's mass flux is ~0.1, and the ambient current (flowing again from right to left) is moderate, the steady outflow displays *multiple equilibria* as shown by the cross-hatched area in Figure 6a. Namely, for some

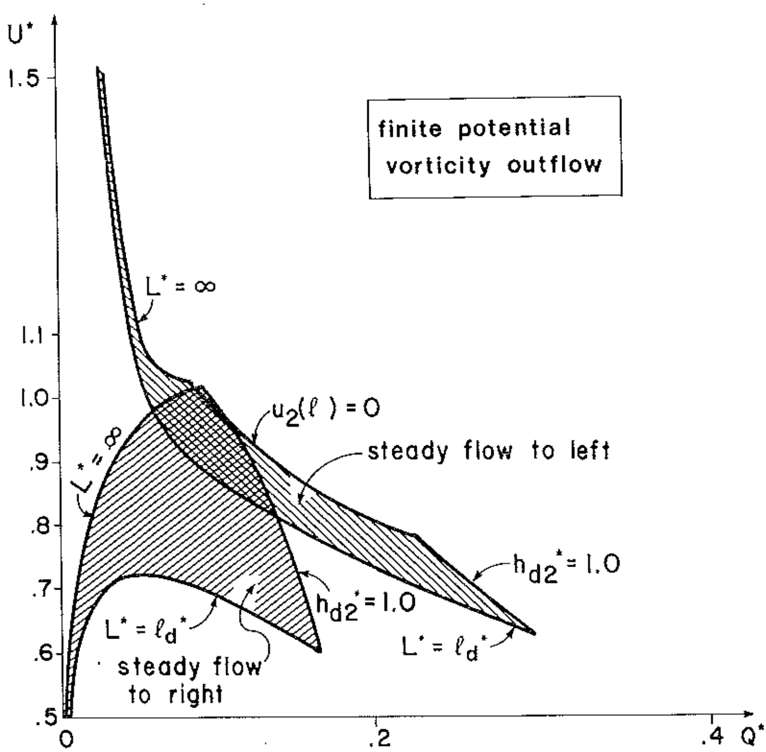

Figure 6a The various flows regimes as a function of the ambient fluid speed (U^*) and the outflow's discharge (Q^*) for a finite (nonzero) potential vorticity ($K=\frac{1}{2}$). Shaded areas denote the existence of steady solutions. Note that deflections to either the right or left are possible. Furthermore, the cross-hatched area denotes the existence of *multiple equilibria*. For the detailed solution see Figures 6b and 6c.

range of parameters, the outflow can deflect to either the right or left for the *same* discharge and ambient flow. It is, of course, possible that one of these states is unstable. A brief qualitative examination of the stability question indicates, however, that there are no significant differences between the two states and that the flows are stable. A thorough, detailed examination of the stability question is beyond the scope of this study.

As before, to determine the solution graphically [for a given ambient flow (U^*) and a given outflow mass flux Q^*] one begins by determining l_d^* from the upper left panel of Figure 6b. With this numerical value one proceeds to the remaining panels in Figure 6b to determine the near coast depth ($\hat{h}_{d2}^*$), the near coast speed [$u_{2d}^*(0)$] and the corresponding channel width (L^*).

At this stage it is appropriate to examine the validity of our assumption regarding the negligible contribution of the source to the integrated momentum equation. To do so, consider the x-momentum equation for a source with a finite width (i.e. a channel) instead of a point source (Figure 4)

$$u_2 \partial u_2/\partial x + v_2 \partial u_2/\partial y - f u_2 + g\partial\eta/\partial x + g'\partial h_2/\partial x = 0.$$

Multiplication of this equation by h and integration over the area shown by the dashed line in Figure 4 gives the balance of forces in the x direction,

$$-\oint h_2 u_2^2\, dy + \oint h_2 u_2 v_2\, dx + f\oint \psi_2\, dy + \tfrac{1}{2}g'\oint h_2^2\, dy$$
$$+ g\iint h_2(\partial\eta/\partial x)\, dx\, dy = 0,$$

where the line integrals are taken counter-clockwise.

The direct contribution of the distributed source to the x component of the flow-force comes from the second term,

$$\int_{-\delta}^{\delta} h_2 u_2 v_2\, dx,$$

(where 2δ is the feeding channel width), which has been taken to be zero in deriving (4.7). It is not *a priori* obvious that this integral vanishes in the limit $\delta \to 0$ because, as $\delta \to 0$, u_2, $v_2 \to \infty$. We shall show, however, that in our case, the integral does vanish when $\delta \to 0$. To show this note that, when the width is *finite* (i.e. $\delta \neq 0$), the integral

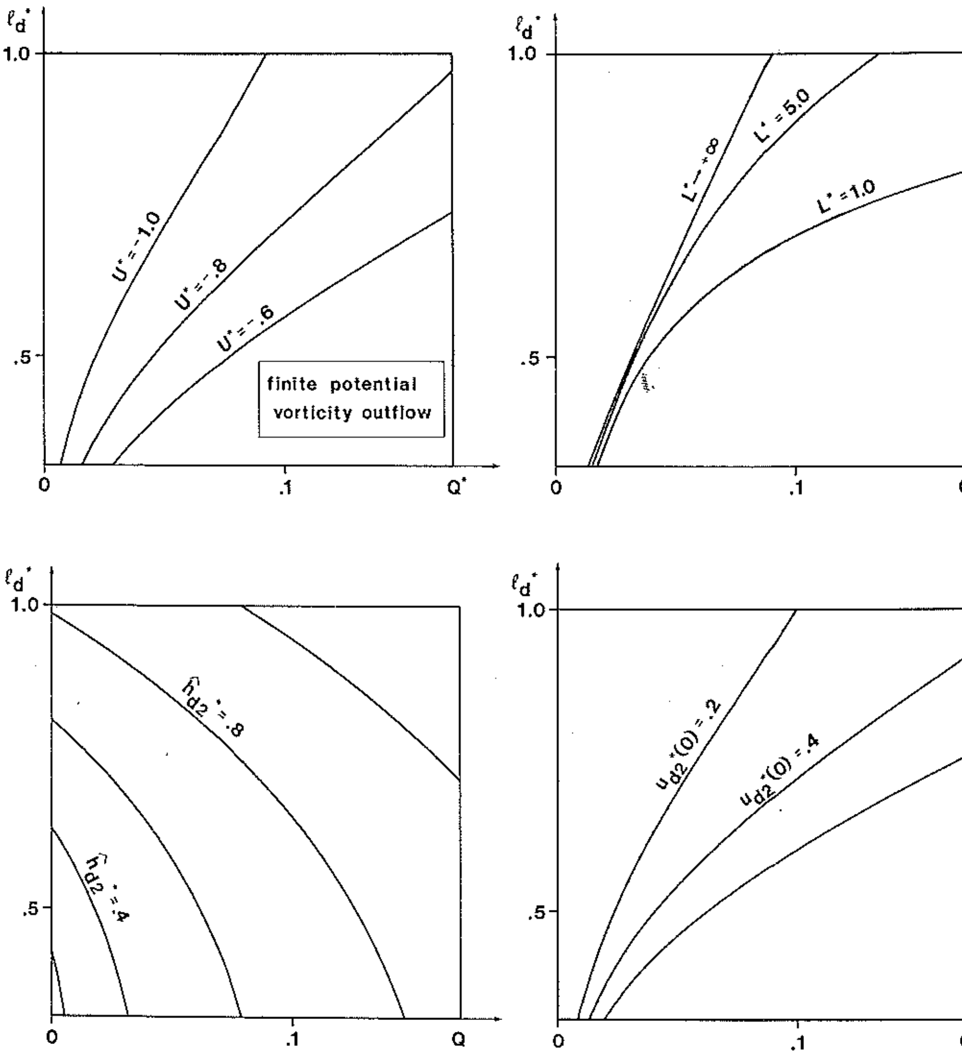

Figure 6b The solution for a finite potential vorticity outflow ($K = \frac{1}{2}$) that deflects to the *right*. (The solution for outflows that deflects to the left is shown in Figure 6c.) First, one determines the outflow's width (l_d^*) from the ambient speed U^* and the outflow's discharge Q^* (upper left). Then, one proceeds to determine the near wall depth $\hat{h}_{d2}$ (lower left), the channel width (upper right), and the near wall speed (lower right).

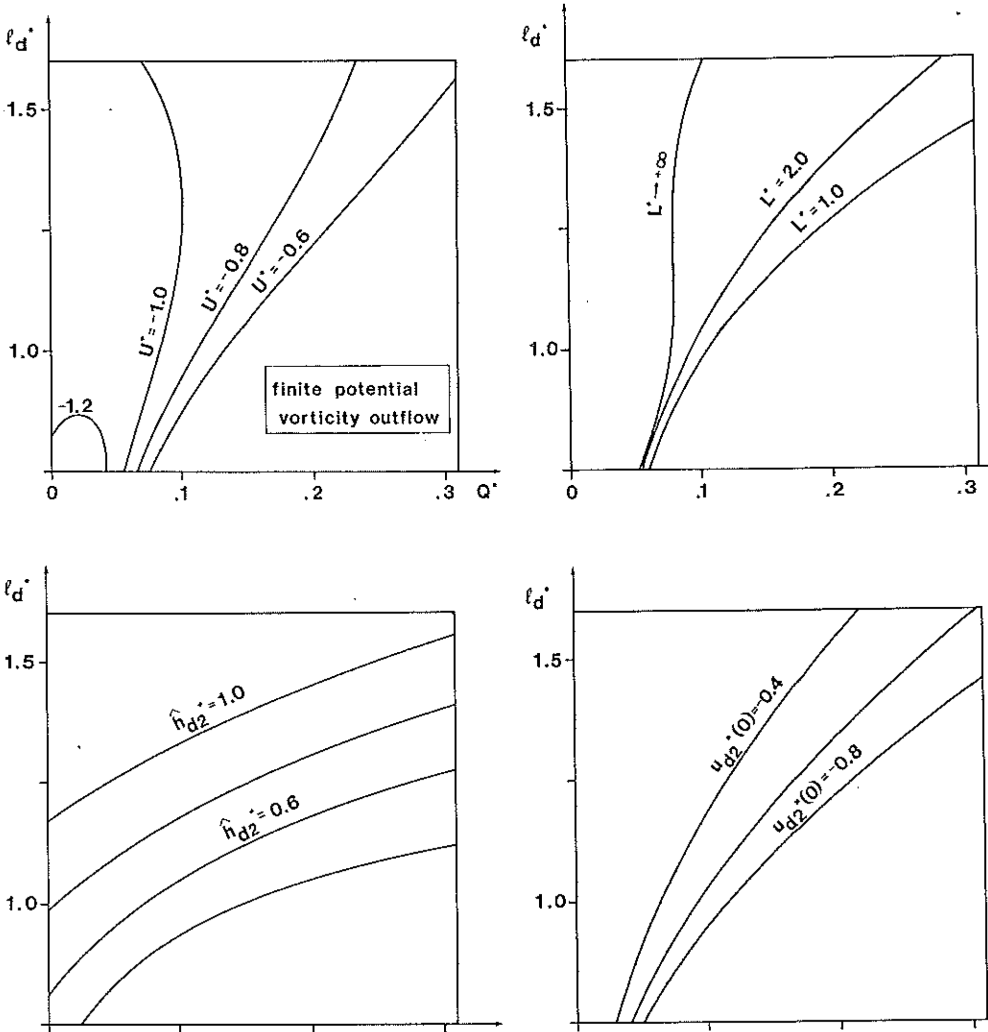

Figure 6c The same as Figure 6b but for the regime that is associated with a deflection to the *left*.

vanishes if the flow is *symmetrical*, i.e. $h_2(x)=h_2(-x)$; $v_2(x)=v_2(-x)$ and $u_2(x)=-u_2(-x)$. It is well known that in the absence of rotation (i.e. $f=0$), the flow is always symmetrical. Namely, for $f=0$ the integral

$$\int_{-\delta}^{\delta} h_2 u_2 v_2 \, dx$$

vanishes whether or not $\delta \to 0$. Since the importance of rotation is measured by U/fl (where U and l are the velocity and length scales) and at the source $U \to \infty$ and $l \to 0$, it follows that rotation will not be important at the source. Consequently, it is expected that

$$\lim_{\delta \to 0} \int_{-\delta}^{\delta} h_2 u_2 v_2 \, dx = 0,$$

even for $f \neq 0$, as stated in Section 4.

8. SUMMARY

Before listing our conclusions, it is appropriate to stress that our study is based on the assumptions that energy loss, friction and diffusion are small and can be neglected. The results of the study can be summarized as follows:

(1) When light (or heavy) fluid is steadily released near the coast of a resting, infinitely deep and infinitely broad ocean, the flow in the vicinity of the source (i.e. within a few deformation radii) *can never* reach a steady state (Figure 1). This is also the case when the infinitely deep and infinitely broad ocean contains a long-shore current.

(2) The above steady states are impossible because the form-drag exerted on the outflow by the ambient fluid cannot balance the flow-force associated with the outflow.

(3) Steady outflows states are possible only when the ocean is of finite depth and there is a long-shore current.

(4) For such flows, the results are sensitive to changes in the distribution of potential vorticity but are not very sensitive to

changes in the other physical parameters such as the ambient fluid speed.

(5) The steady solution for a zero potential vorticity outflow corresponds to a deflection to the right (looking off-shore) and an ambient current flowing from left to right or right to left (Figure 5).

(6) In contrast to a zero potential vorticity outflow, an outflow with a finite potential vorticity can deflect to either left (Figure 6) or right depending on its mass flux and the ambient fluid speed. For both cases the ambient fluid must be flowing from *right* to *left*. Surprisingly, when the outflow mass flux is *weak* [i.e. $Q^* \sim O(0.1)$] and the ambient current is of $O(1)$, the outflow displays *multiple equilibria*. That is to say, for some range of parameters, the outflow can deflect to either the right or left for the *same* ambient flow and discharge. This means that the outflow can flip from one state to another without any change in the discharge or ambient current.

The above results have applications to various outflows in the ocean. Many rivers empty into semi-enclosed basins with some kind of a mean flow. For instance, the Yangtze River empties into the semi-enclosed East China Sea (see Beardsley *et al.*, 1985), and the Fraser River debouchés into the channel-like Georgia Strait.

Acknowledgements

Computations and derivations using the "Macsyma" were done by Steve VanGorder whose expertise is indispensable. The study was supported by the Office of Naval Research, contract number N00014-82-C-0404.

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