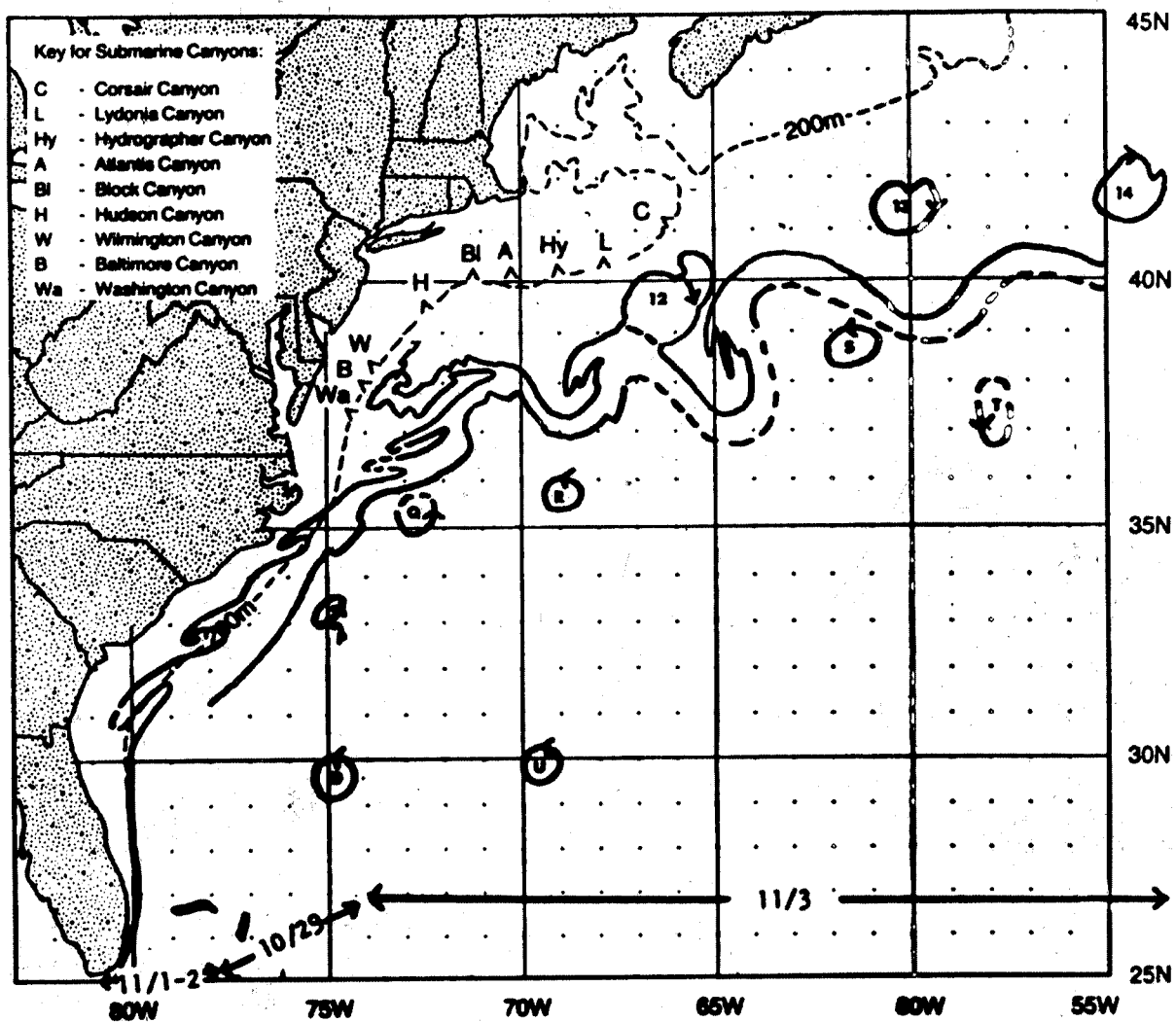


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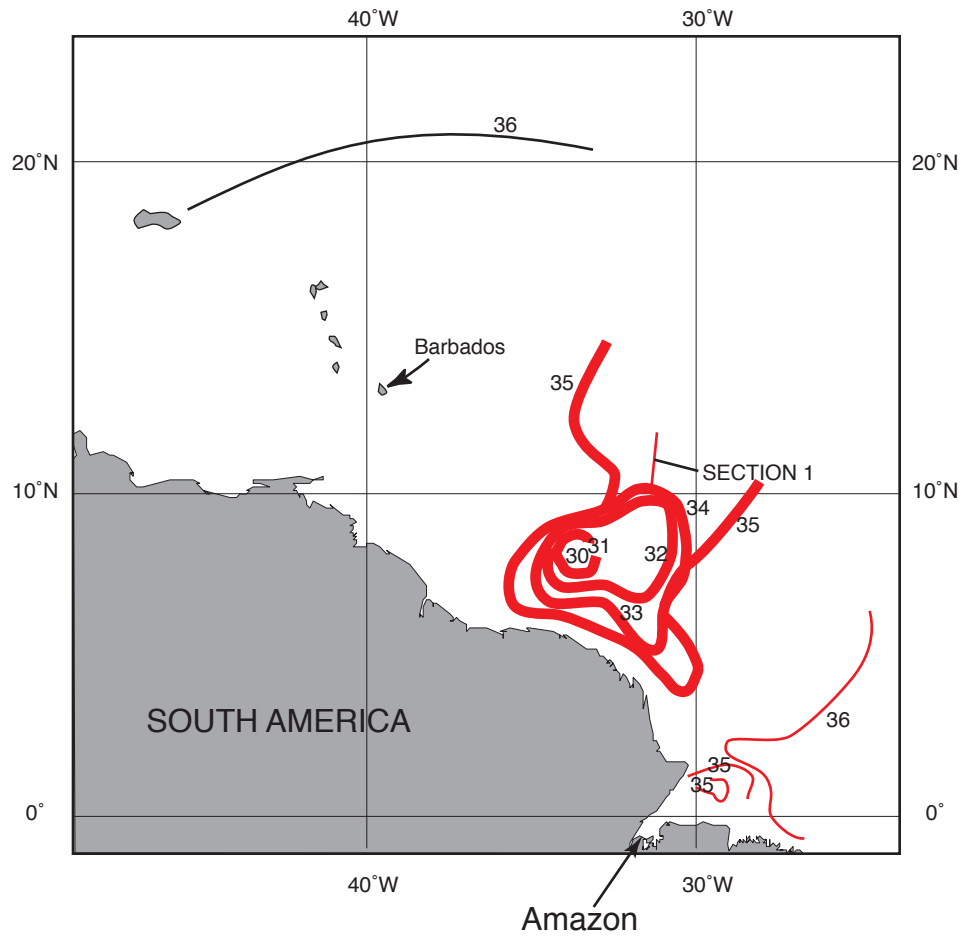
RINGS AND EDDIES AS HEAT EXCHANGE MECHANISM

Course Objectives

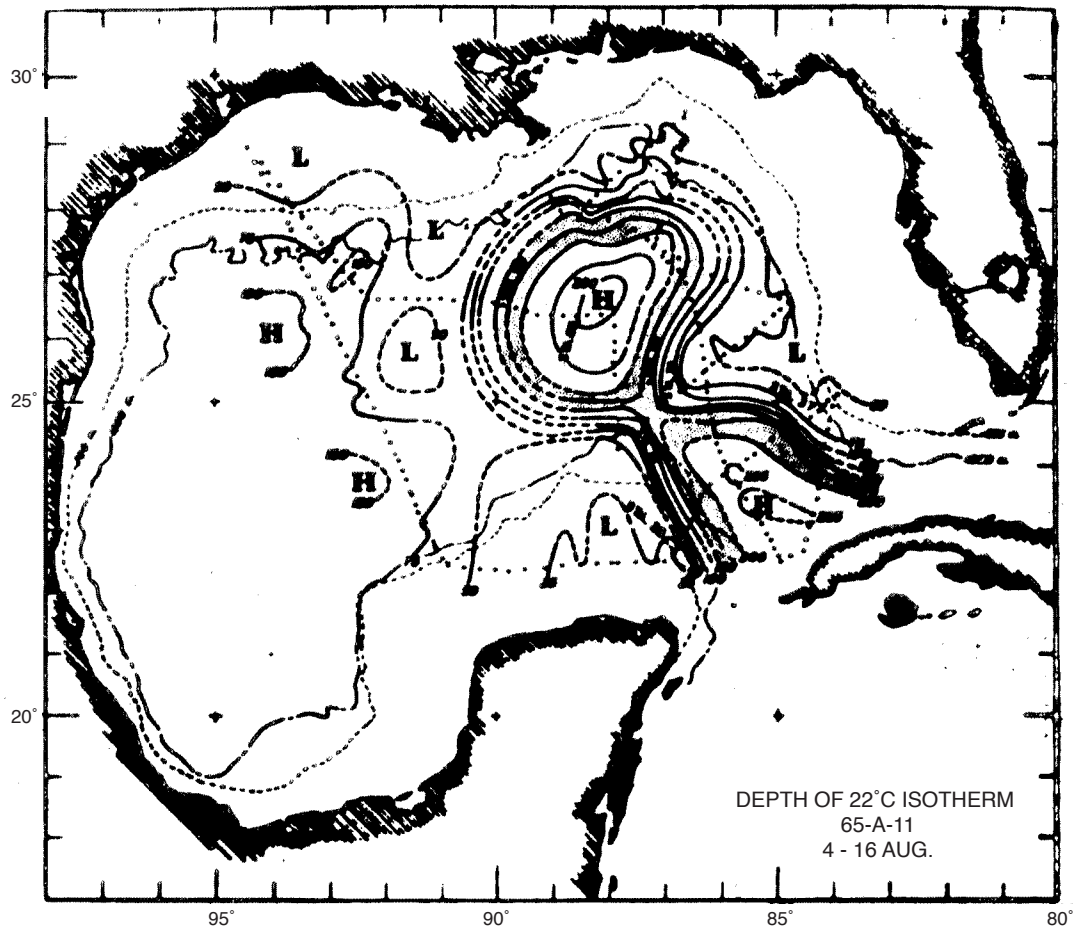
1. To provide a theoretical and observational background for the transfer of heat and salt by eddies and rings
2. To introduce the student to new discoveries and new theories in dynamical oceanography



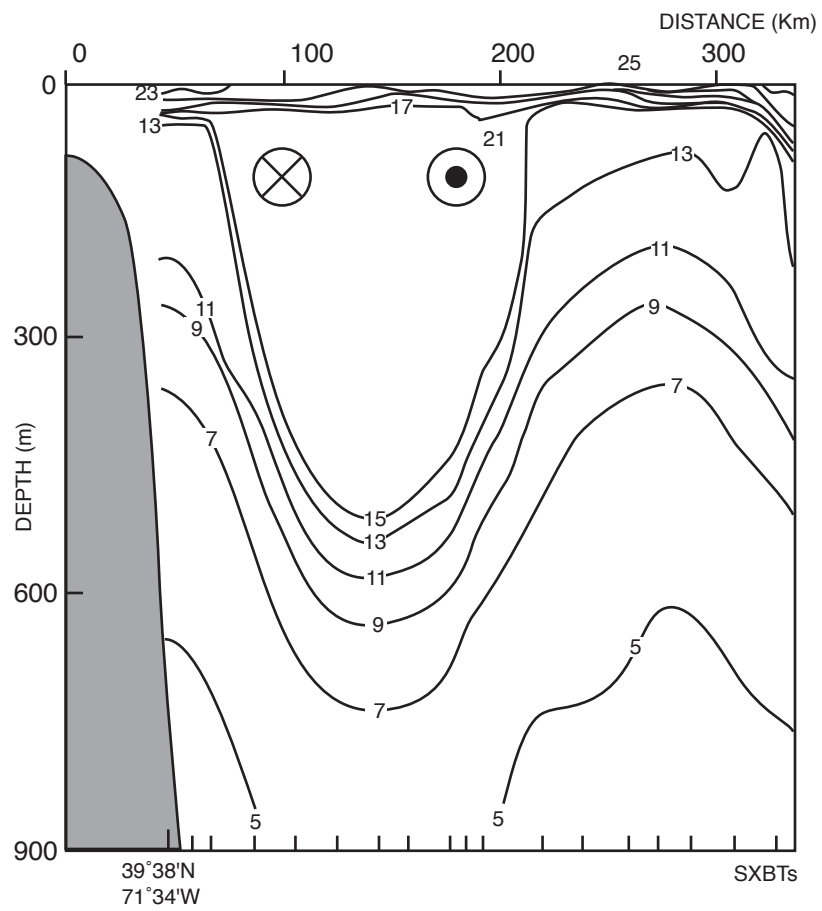
A typical structure of the Gulf Stream and its rings. Note that there are typically more cyclones (cold-core rings) than anticyclones (warm-core rings).



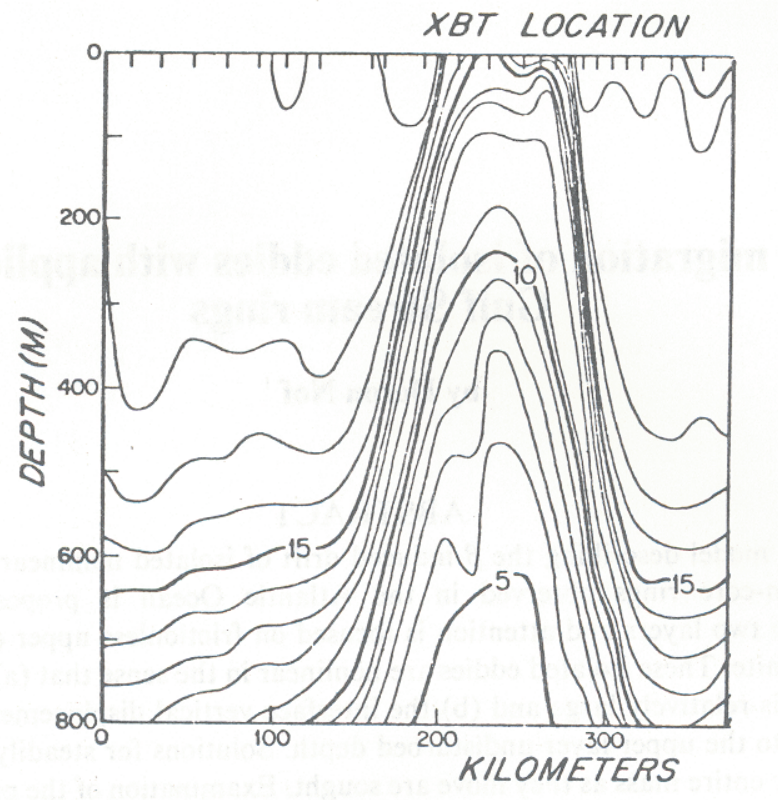
Salinity contours of the Amazon (Dec 1964) showing an Amazonian anticyclone.



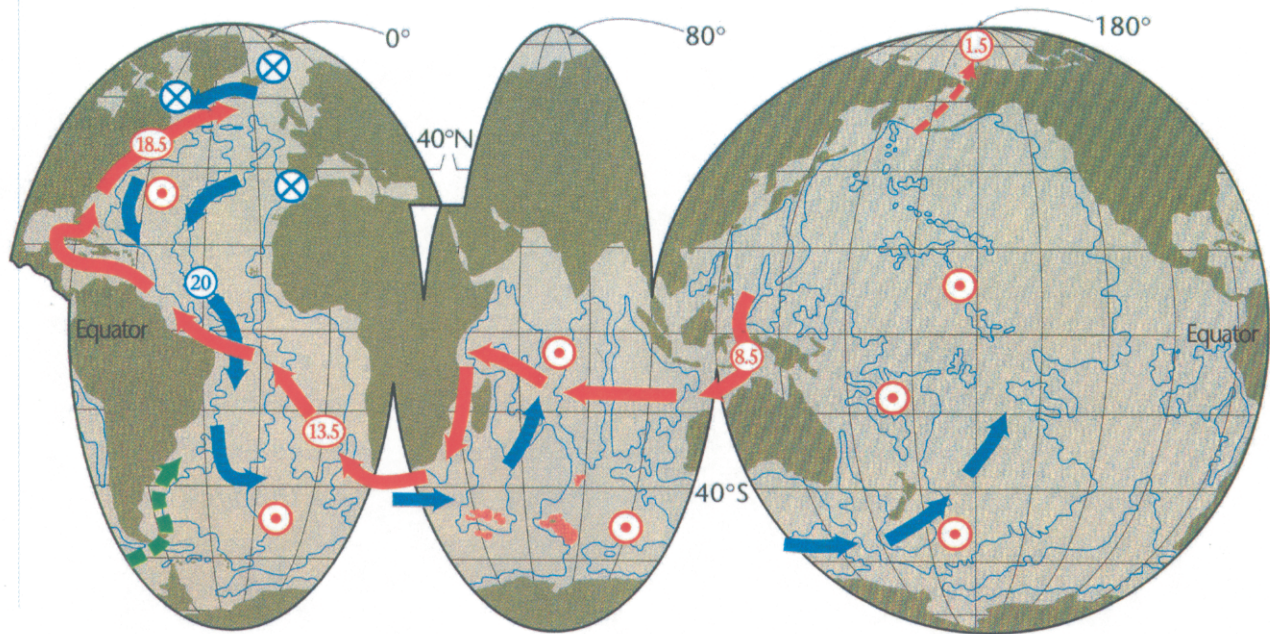
Topography of the 23°C isothermal surface (August 1986) in the Gulf of Mexico showing a ring ready to separate from the current.



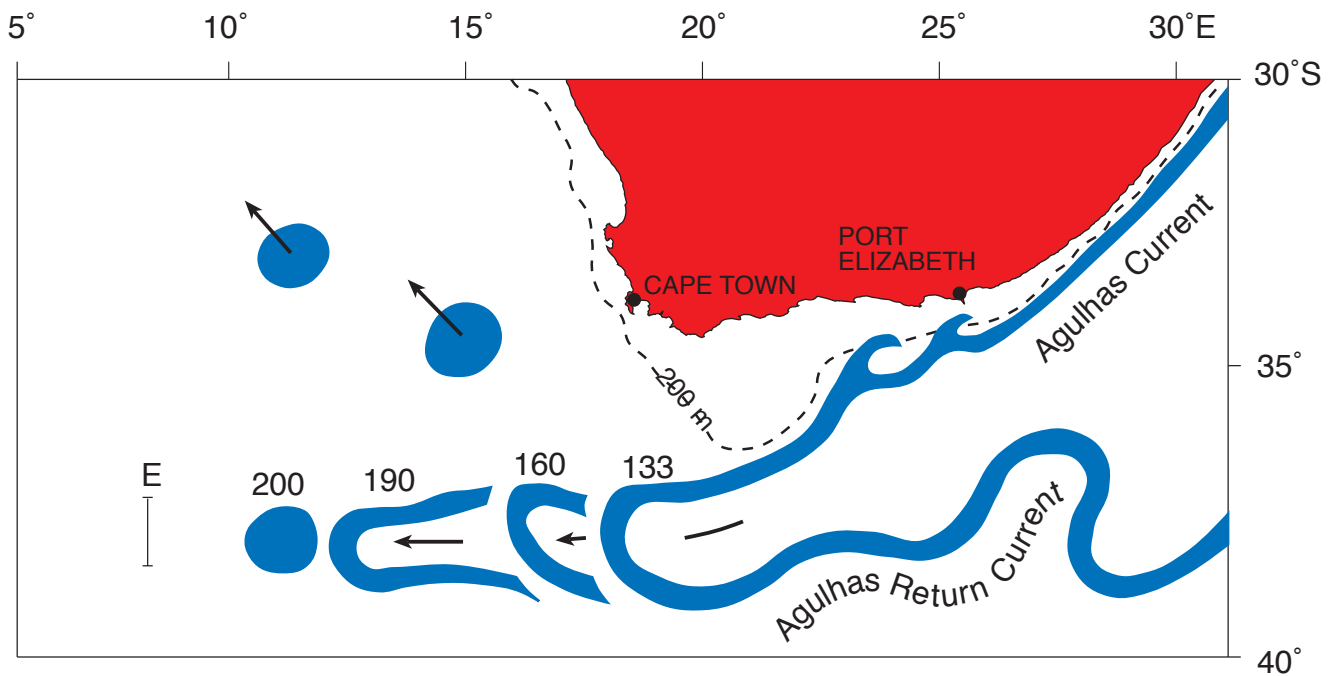
A cross-section through a typical (Gulf Stream) warm-core ring.



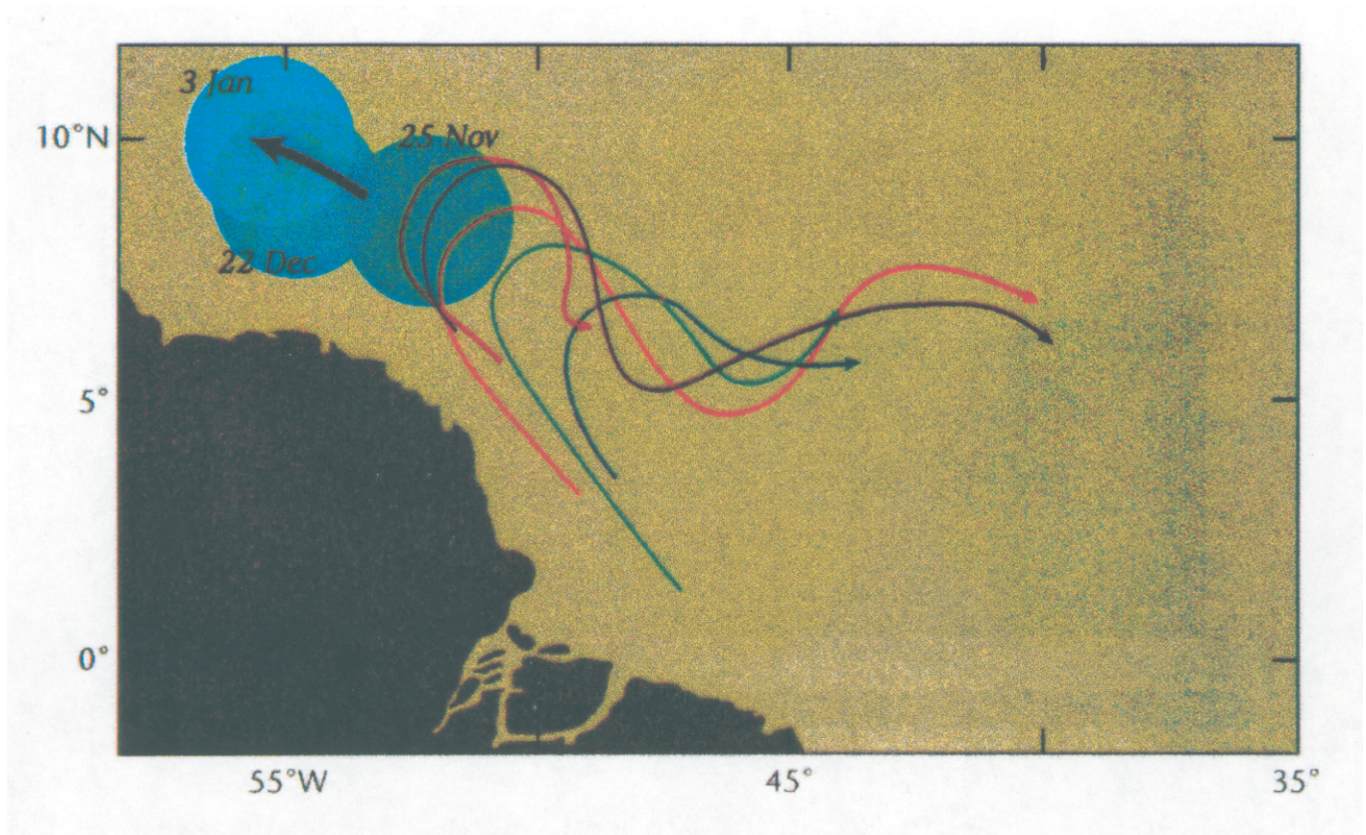
A cross-section through a typical (Gulf Stream) cold-core ring.



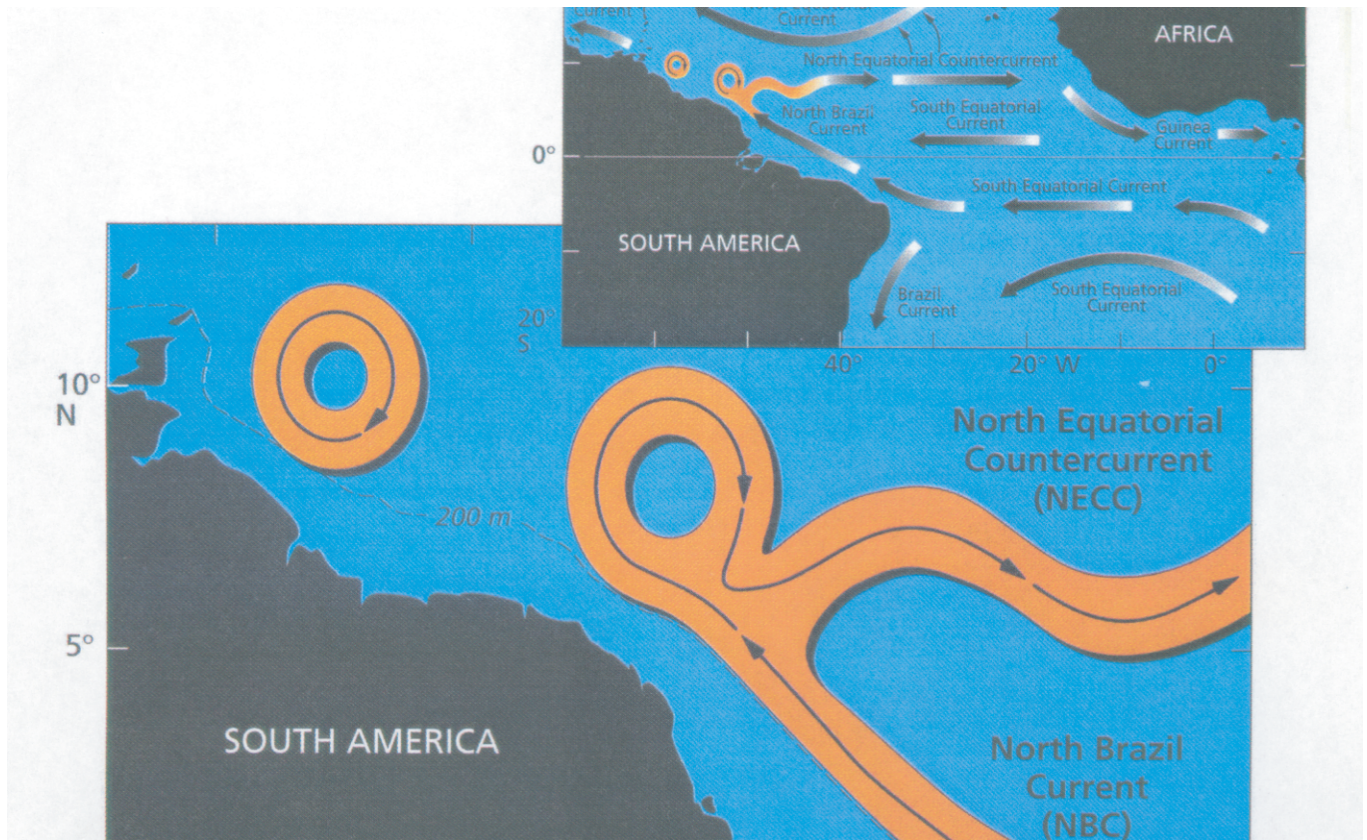
A map of the pathways and transports for NADW formation and renewal, adapted and modified from the "conveyor" circulation of Gordon (1986) by Schmitz (1995). Transport in circles in Sverdrups. As shown in detail in the following two figures, the transport along two crucial points (the tip of Africa and the sub-equatorial Atlantic) occurs via rings.



The establishment of (typical) Agulhas rings which play a crucial role in the path of the water to the NADW formation site.



North Brazil Current retroflexion eddies (and associated surface drifter tracks, 1979). As Agulhas rigs, these rings are important to the so-called great global conveyor structure.



Large-scale schematic diagram for North Brazil Current retroflexion eddies and their environment.

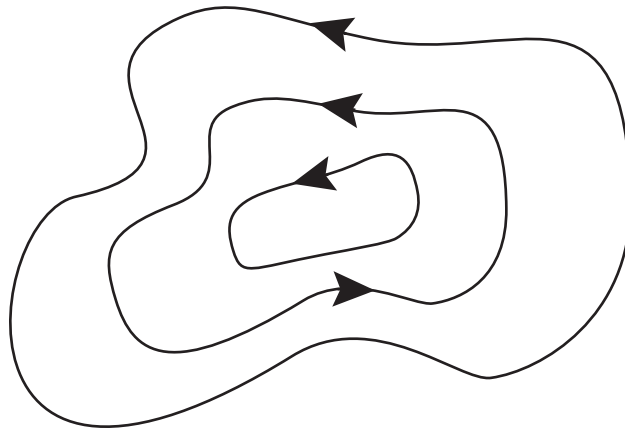
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RINGS AND EDDIES AS HEAT EXCHANGE MECHANISM

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I. Definition of Eddies and Rings

The term “eddies” is used with regard to a variety of situations in the ocean. For instance, any perturbations on some mean flow are sometimes called “eddies.” In this course we shall deal primarily with **isolated eddies**, which are defined here as features which have **closed streamlines**. (By “**closed streamlines**” we mean that the fluid is circulating within the feature as shown schematically below.) They are not necessarily connected to any mean flow (although they may result from one).



They are, to some degree, analogous to hurricanes and tornadoes. For example, Gulf Stream rings [see special *Oceanus* issue and Csanaday (1979) JGR paper], Agulhas rings and North Brazil Current rings have **closed streamlines**. Note that the term “rings” is usually used with regard to **specific** eddies, e.g., Gulf Stream or North Brazil Current rings.

II. Why are eddies important?

- (a) They are very **energetic**. Typical orbital speeds $\sim 1\text{--}2 \text{ m sec}^{-1}$, relatively **large** ($\sim 100 \text{ km}$) and **deep** ($\sim 500 \text{ m}$).

- (b) They move in the ocean so that they carry properties from one region to another.
- (c) Aside from their energy, they carry anomalous properties such as **salinity** or **temperature**. So, they are important to the **heat** and **salt fluxes** within the ocean.

In the coming weeks we shall talk about the **movements** of the eddies. There are three kinds of movements. One is the movement due to β , the variation of the Coriolis parameter with latitude. The second, advection by some mean flow. The third, mutual advection of adjacent eddies. We shall first discuss the movements due to β . The first one who recognized that vortices (eddies) must move due to β was Rossby (1948).

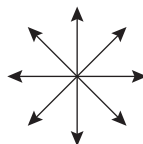
Homework #1:

Draw the path of a particle on a mid-latitude β -plane. Namely, consider a ball on a β -plane obeying,

$$du/dt - fv = 0 \quad \text{where } f = f_0 + \beta y$$

$$dv/dt + fu = 0$$

Note that there is no analytical solution. Take the initial speeds to be due to the 8th directions shown below,



(Ref. Cushman-Roisin 1982 GAFD.)

Note that a particle on a sloping bottom on an f-plane obeys,

$$du/dt + f_0 v = 0$$

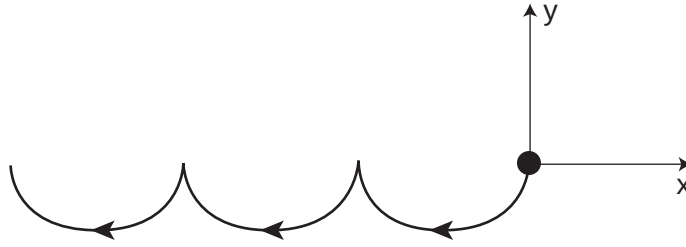
$$dv/dt = -f_0 u - gS$$

which has the exact analytical solution

$$X = -\frac{gS}{f_0^2} (1 - \cos ft)$$

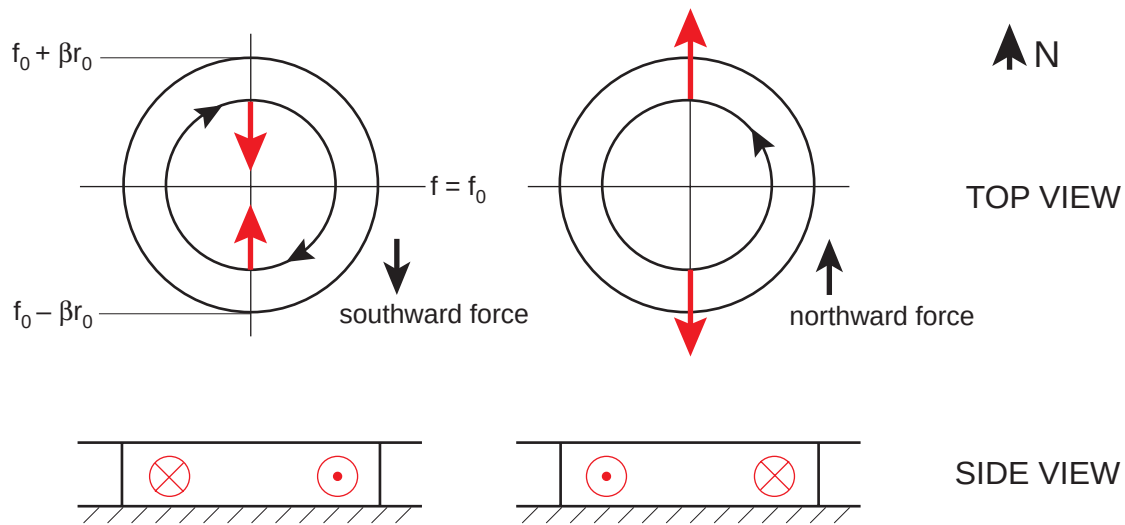
$$Y = \frac{gS}{f_0^2} (ft - \sin ft) ,$$

where X and Y are the positions of the particle at time t. This solution corresponds to a trajectory with cusps as shown below,

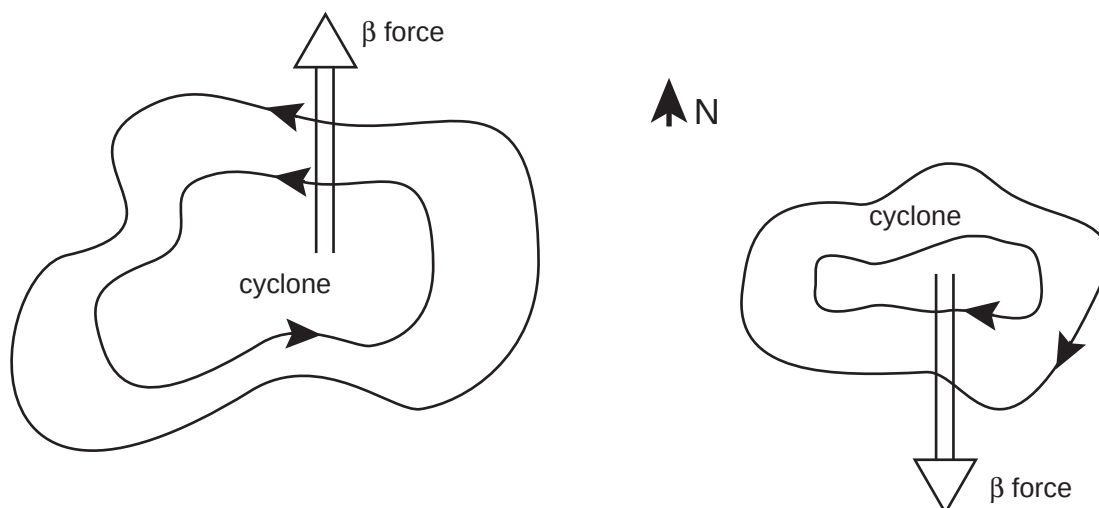


III. General properties of eddies

Rossby (1948) recognized that on any **circular** streamline there should be a net meridional force due to β . To see this, note that if we take the orbital speed to be the same for all points along the circle, the Coriolis force (red) is greater along the northern part of the vortex.



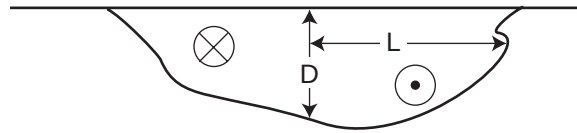
As a result, an anticyclone (cyclone) will move southward (northward) once β is “turned on.” Rossby further concluded that, because of a combined Coriolis force acting on the entire vortex, anticyclones (cyclones) will ultimately move westward (eastward). This is nice but it is not very clear that it is really applicable because how do we know that the velocity does not change as we go **along** a streamline. We don’t really know this so we shall do a more careful analysis and show that actually on **any** closed circulation there is a net force. Namely, as shown schematically in the figure below, even if the velocity is not constant and even if the shape is not circular, there will be a northward (southward) force on a cyclone (anticyclone).



We shall also see that, although Rossby's conclusion that anticyclones move westward is correct, his conclusion that cyclones move eastward is incorrect. This is due to an additional external force which he neglected. It will take us a while to show this and we will need to work with the **shallow water equations**. So we will review and derive them first.

IV. Shallow water equations

These will be used throughout the course. Talk about **steady** state and **frictionless** fluid. Friction is not important because of the time scale. Namely, if we have an eddy with a diameter L and depth D , then the time required for the penetration of friction to its core is $t \sim O\left(\frac{L^2}{\nu_H}\right)$, where ν_H is the horizontal eddy viscosity, or $t \sim O\left(\frac{D^2}{\nu_v}\right)$, where ν_v is the vertical eddy diffusivity.



We can take $L \sim 100$ km and $\nu_{H\Box} \sim O(10^6 \text{ cm}^2 \text{ s}^{-1})$, but some people will argue that $10^6 \text{ cm}^2 \text{ s}^{-1}$ is either too small or too big. We find,

$$t \sim O\left(\frac{10^{14}}{10^6}\right) = 10^8 \text{ sec} \sim \boxed{3 \text{ years}} .$$

So we are o.k. if we talk about eddies which are, say, 6 months old. Actually, the three years' estimate is probably relatively small because there are indications that eddies which do not collide with a boundary or another current may survive even for 5–10 years

without much decay. (Note that the vertical eddy diffusivity gives a much longer time scale.)

The general **momentum** equations are,

$$uu_x + vu_y + wu_z + 2\Omega \cos\theta w - 2\Omega \sin\theta v = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (4.1)$$

$$uv_x + vv_y + wv_z + 2\Omega \sin\theta u = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (4.2)$$

$$uw_x + vw_y + ww_z + 2u\Omega \cos\theta + g = -\frac{1}{\rho} \frac{\partial p}{\partial z} . \quad (4.3)$$

and the conservation of mass is

$$u_x + v_y + w_z = 0 , \quad (4.4)$$

i.e., the fluid is incompressible and has a constant density ρ .

First make the **hydrostatic** approximation,

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g \quad (4.5)$$

This assumption is correct as long as two conditions are satisfied. First, we require,

$$fU \ll g$$

which is always satisfied because

$$U \sim 1 \text{ m sec}^{-1} \quad \text{and} \quad f \sim 10^{-4} \text{ sec}^{-1} .$$

Second, we must also require:

$$U \frac{W}{L} \ll g ,$$

which, in view of the continuity equation which implies,

$$\frac{W}{U} \sim O\left(\frac{D}{L}\right) ,$$

gives,

$$U^2 \frac{D}{L^2} \ll g$$

or

$$\boxed{\frac{U^2}{gD} \left(\frac{D}{L}\right)^2 \ll 1}$$

The term $\frac{U^2}{gD}$ is sometimes called the Froude number (Fr). In the ocean, $D \sim O(10000 \text{ m})$ and $L \sim O(100,000 \text{ m})$ so we are o.k. Note, however, that in the lab we need to require $Fr \ll 1$ because $D \sim O(L)$ [since in the ocean $D \ll L$, we are o.k. even if $Fr \sim O(1)$].

Next, we shall look at the **fourth term** in (4.1). We require that

$$2\Omega \cos\theta w \ll 2\Omega \sin\theta v ,$$

which is usually o.k. but may not be correct near the equator.

With the above assumptions the governing equations reduce to:

$$uu_x + vu_y + wu_z - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (4.6a)$$

$$uv_x + vv_y + wv_z + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (4.6b)$$

$$u_x + v_y + w_z = 0 . \quad (4.6c)$$

Next, integrate the hydrostatic pressure (4.5) in z to get,

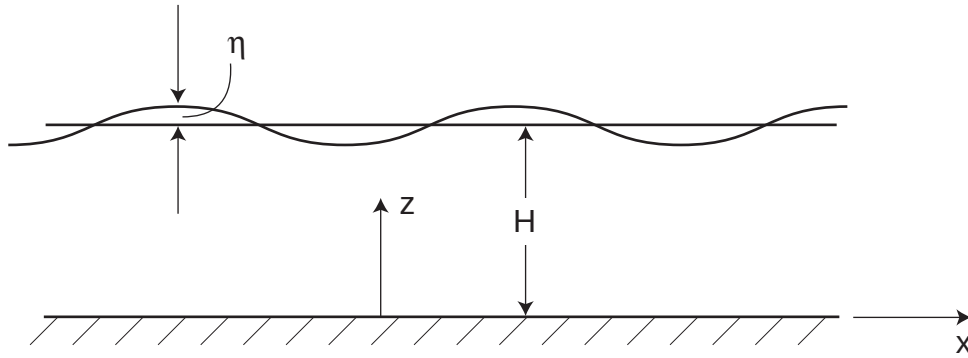
$$p = -\rho gz + P'(x,y) ,$$

where P' is a function of x and y alone. Since

$$\frac{\partial p}{\partial x} = \frac{\partial P'}{\partial x} \quad ; \quad \frac{\partial p}{\partial y} = \frac{\partial P'}{\partial y} \quad ,$$

it follows that the right-hand side of (4.6ab) is a function of **x and y alone**. Therefore, the **left-hand side** must also be a function of x,y alone; i.e., $u \neq f(z)$, $v \neq f(z)$. Hence, the **third** terms on the left-hand side (wu_z and wv_z) can be neglected. Note that **we could not do this by scaling and that w is small but not zero**.

Consider now a barotropic fluid with a free surface and flat bottom:



and note that, under these conditions, the pressure at z is,

$$P = \rho g (\eta + H - z) \quad \text{so that,}$$

$$P' = g\rho(\eta + H) \quad .$$

Under the above hydrostatic conditions the equations are,

$$\boxed{uu_x + vu_y - fv = -g\eta_x} \quad (4.7a)$$

$$\boxed{uv_x + vv_y + fu = -g\eta_y} \quad (4.7b)$$

$$u_x + v_y + w_z = 0 \quad . \quad (4.7c)$$

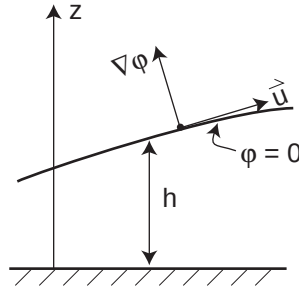
Now, it is convenient to integrate the continuity equation from the bottom to the top.

$$\int_0^{H+\eta} u_x dz + \int_0^{H+\eta} v_y dz + \int_0^{H+\eta} w_z dz = 0 \quad . \quad (4.8)$$

Since $u = u(x,y)$ and $v = v(x,y)$, it follows that (4.8) can be written as,

$$(H + \eta)u_x + (H + \eta)v_y + w|_{H+\eta} - w|_0 = 0 . \quad (4.9)$$

Recall now the condition of no-normal flow through the free surface:



$$\nabla\phi \cdot \mathbf{u} = 0 \quad ; \quad \phi = h - z = 0$$

i.e.,

$$u\phi_x + v\phi_y + w\phi_z = 0$$

$$uh_x + vh_y - w = 0$$

so that

$$w_{\text{top}} = uh_x + vh_y$$

and

$$w_b = 0 .$$

With these conditions the continuity equation (4.9) becomes,

$$hu_x + hv_y + uh_x + vh_y = 0$$

or

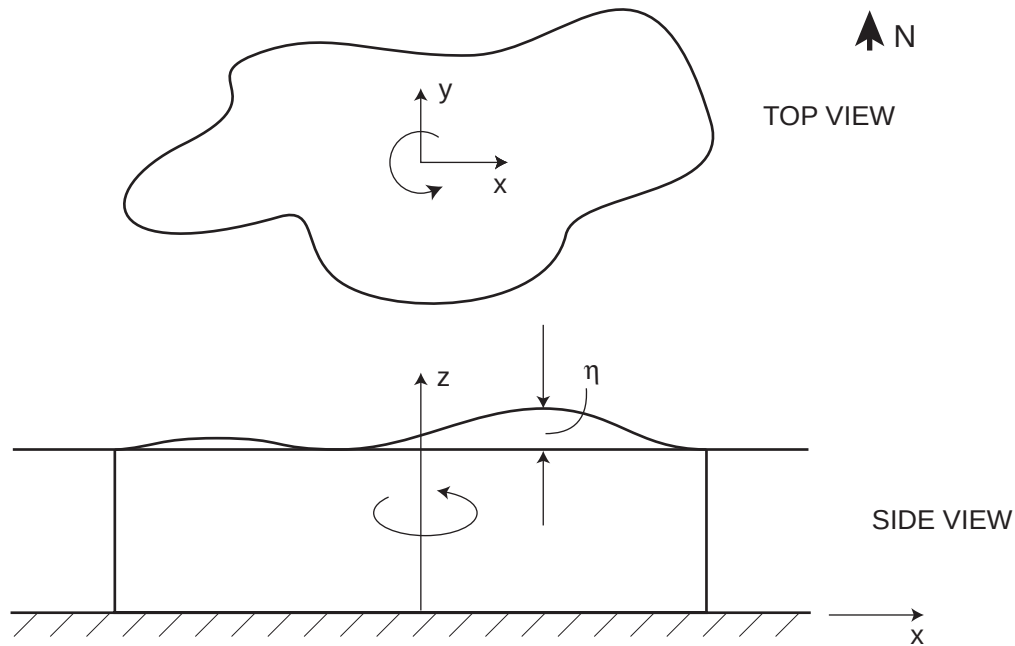
$$\boxed{(hu)_x + (hv)_y = 0 .} \quad (4.10)$$

This completes the derivation of the shallow water equations (4.7a), (4.7b) and (4.10).

Note that, in the above case η is not necessarily small compared to H !

V. The stationarity condition

Consider an isolated patch on a β plane:



We would like to find whether or not there is a **net force** acting on the patch. For this case, we shall take $\eta \ll H$ because from geostrophy,

$$\eta \sim O(fUL/g) \sim 10^{-4} \propto 1 \propto \frac{100,000}{10} \sim 1 \text{ m}$$

whereas

$$H \sim 5000 \text{ m} .$$

This means that we can use the **rigid lid approximation** which implies that the full continuity equation,

$$\frac{\partial}{\partial x} [(H + \eta)u] + \frac{\partial}{\partial y} [(H + \eta)v] = 0$$

can be approximated by

$$u_x + v_y = 0 ,$$

[because $\partial H / \partial x = 0$ and $\eta \ll H$].

It is important to note that, with the rigid lid approximation, **we neglect η in the continuity equation but not in the pressure terms!**

Then, we have

$$uu_x + vu_y - fv = -g\eta_x$$

$$uv_x + vv_y - fu = -g\eta_y$$

$$u_x + v_y = 0 .$$

Recall that, here, in the rigid-lid barotropic case both w and w_z are taken to be zero.

To obtain the net meridional force or what we shall later call the “stationarity condition,” integrate the momentum equation in the y direction over the patch:

$$\int_V \int_V \int_V (uv_x + vv_y) dx dy dz + \int_V \int_V \int_V (f_0 + \beta y) u dx dy dz = -g \int_V \int_V \int_V \frac{\partial \eta}{\partial y} dx dy dz . \quad (5.1)$$

Since $u = u(x,y)$, $v = v(x,y)$, (5.1) can be expressed as

$$H \int_S \int_S (uv_x + vv_y) dx dy + H \int_S \int_S (f_0 + \beta y) u dx dy = -gH \int_S \int_S \frac{\partial \eta}{\partial y} dx dy dz . \quad (5.2)$$

Recall now that we can always write,

$$uv_x + vv_y + v(u_x + v_y) = \frac{\partial}{\partial x} (uv) + \frac{\partial}{\partial y} (v^2) ,$$

so that (5.2) becomes,

$$\int_S \int_S \left[\frac{\partial}{\partial x} (uv) + \frac{\partial}{\partial y} (v^2) \right] dx dy + \int_S \int_S (f_0 + \beta y) u dx dy = -g \int_S \int_S \frac{\partial \eta}{\partial y} dx dy . \quad (5.3)$$

Next, recall Stokes’ theorem,

$$\int_S \int_S \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx dy = \oint N dx + M dy$$

and define a streamfunction ψ such that

$$\psi_y = -u \quad ; \quad \psi_x = v .$$

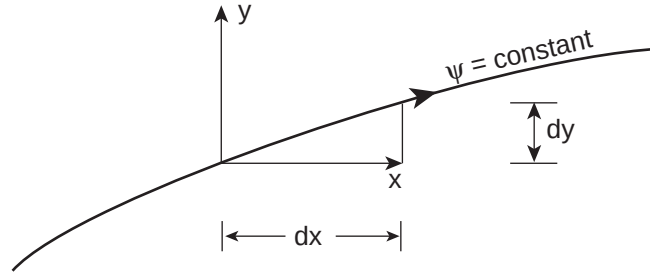
With this definition, the continuity equation is automatically satisfied because

$$u_x + v_y = -\psi_{yx} + \psi_{xy} = 0 .$$

Equation (5.3) now becomes,

$$\oint -v^2 dx + \oint uv dy - \iint_S (f_0 + \beta y) \psi_y dx dy = g \oint \eta dx . \quad (5.4)$$

We now note that around the vortex there is no movement so that $\eta = 0$ along the boundary. Also, the outermost line is a streamline so that along it $u dy = v dx$. To see this, note that



$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \text{ along a streamline}$$

so that $v dx = u dy$.

In view of these, (5.4) takes the form:

$$-f_0 \oint \psi dx - \beta \iint_S \left[\frac{\partial}{\partial y} (y\psi) - \psi \right] dx dy = 0 . \quad (5.5)$$

Since ψ is a constant along the boundary, (5.5) takes the form,

$$-\beta \iint_S \frac{\partial}{\partial y} (y\psi) dx dy + \beta \iint_S \psi dx dy = 0 \quad (5.6)$$

or

$$\beta \oint y\psi dx + \beta \iint_S \psi dx dy = 0 \quad (5.7)$$

Next, define ψ to be zero along the boundary so that (5.7) becomes the desired stationarity condition:

$$\boxed{\beta \iint_S \psi \, dx dy = 0 .} \quad (5.8)$$

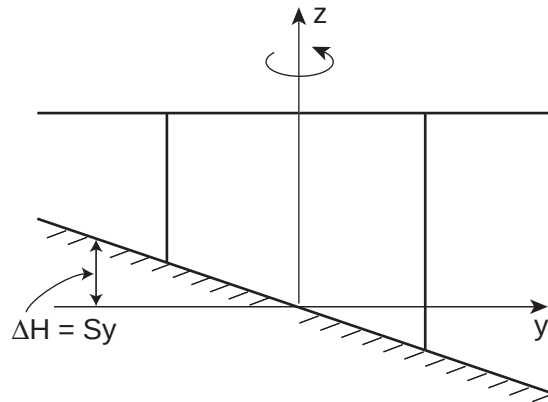
Note that (5.8) is nothing more than an expression for the net southward force. To see this, note that it can be traced to the integral of the differential southward Coriolis term (f_u). Hence, a patch cannot be stationary on a β plane unless $\beta \iint_S \psi \, dx dy = 0$, implying

that the average value of ψ must be zero. For instance, a double cell feature can be stationary. (We shall later refer to such a feature as a “Mondon.”)

To derive the detailed solution of such a feature it will be necessary for us to use the Bernoulli integral and the potential vorticity equation. This will be done in the next section.

Homework #2

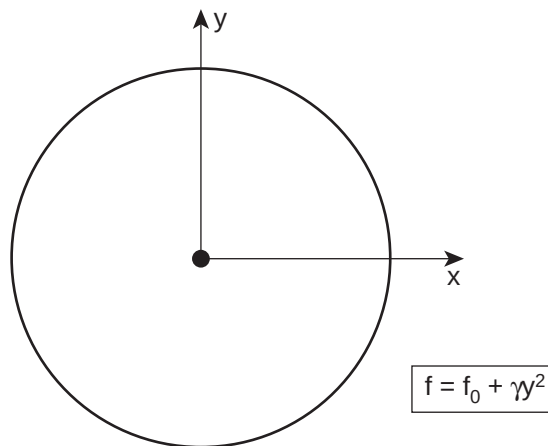
1. Determine the stationarity condition for an isolated barotropic eddy resting on a sloping bottom on an f plane.



2. Suppose that on some non-spherical planet the Coriolis force is given by

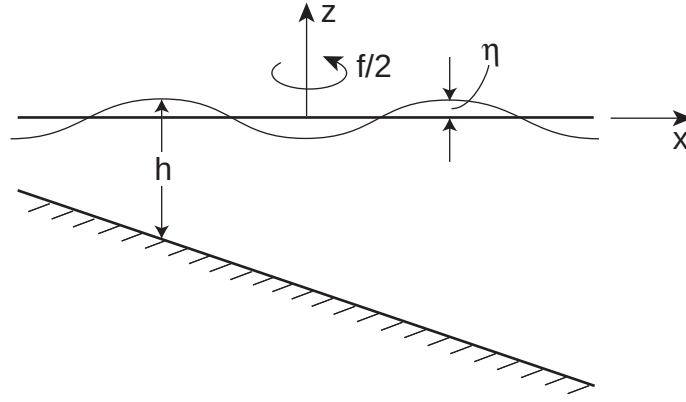
$$f = f_0 + \gamma y^2 \quad \text{where } \gamma = \text{constant}.$$

Discuss the expected behavior of a barotropic eddy due to the variation of the Coriolis parameter with y . Assume that the eddy is located such that its center is at $y = 0$.



VI. The Bernoulli integral and potential vorticity equation

Consider a steady barotropic fluid with a sloping bottom,



$$uu_x + vu_y - fv = -g \eta_x \quad (6.1)$$

$$uv_x + vv_y + fu = -g \eta_y \quad (6.2)$$

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \quad (6.3)$$

To derive the Bernoulli, multiply (6.1) by uh and (6.2) by vh and add the equations thus obtained,

$$v^2 hu_x + uhvuh - fvuh = -guh\eta_x$$

+

$$vhuv_x + hv^2vy + fvhu = -gvh\eta_y .$$

Also use,

$$-uh = \tilde{\psi}_y \quad ; \quad vh = \tilde{\psi}_x ,$$

where the tilde ($\sim$) is used to distinguish the streamfunction from that related to the constant depth case. It is easy to show that the above result can be written in terms of a Jacobian:

$$\begin{aligned} J \left[\tilde{\psi} ; \frac{u^2 + v^2}{2} + g\eta \right] &= \tilde{\psi}_x (uu_y + vu_y + g\eta_y) - \tilde{\psi}_y (uu_x + vu_x + g\eta_x) \\ &= vh (uu_y + vv_y + g\eta_y) + uh (uu_x + vv_x + g\eta_x) \end{aligned}$$

$$= vhuu_y + v^2hv_y + vhg\eta_y + u^2hu^x + uhvv_x + uhdg\eta_x = 0.$$

Namely,

$$J\left[\tilde{\psi} ; \frac{u^2 + v^2}{2} + g\eta\right] = 0. \quad (6.4)$$

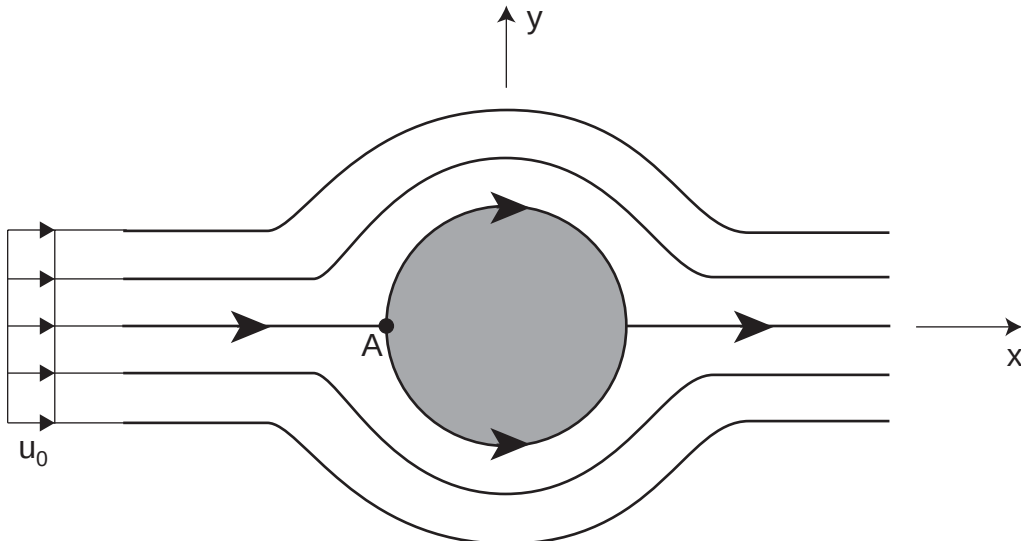
which means,

$$\boxed{\frac{u^2 + v^2}{2} + g\eta = B(\tilde{\psi})}, \quad (6.5)$$

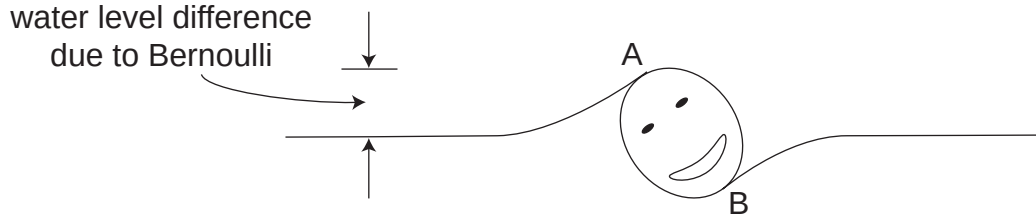
implying that the sum of $(u^2 + v^2)/2 + g\eta$ is **constant** along a streamline.

However, in general, the constant **varies** from one streamline to another. There are two exceptions to this variation from one streamline to the other. The first is the case of no vorticity ($v_x - u_y \equiv 0$) and no rotation ($f \equiv 0$) and the second is the so-called zero potential vorticity case where $v_x - u_y + f = 0$. In both of these cases the Bernoulli function is uniform throughout the field. Why this is so will become apparent shortly.

Although not frequently used, the Bernoulli is a **powerful tool** because it does not involve any derivatives. For example, a flow around a column (shown below) has a stagnation point A where the water is $u_0^2/2g$ higher than the water at $x \rightarrow -\infty$. With the Bernoulli, there is no need to solve for the entire field in order to calculate the above pressure difference.



In the case of a swimmer (shown below), the Bernoulli shows that the water level difference between A and B increases with increasing swimming speed. Namely, the faster the swimmer goes the less he/she needs to turn the head to the side to take a breath of air.



Next, we would like to obtain the **potential vorticity equation**. To do so, take the derivative of (6.5) with respect to y :

$$uu_y + vv_y + g\eta_y = \frac{dB}{d\tilde{\psi}} \frac{\partial \tilde{\psi}}{\partial y} = \frac{dB}{d\tilde{\psi}} (-uh) ,$$

and subtract it from the familiar momentum equation:

$$uv_x + vv_y + fu = -g\eta_y .$$

One finds,

$$u(v_x - u_y) + fu = u h \frac{dB}{d\tilde{\psi}} ,$$

which gives the potential vorticity equation in the form:

$$\boxed{\frac{v_x - u_y + f}{h} = \frac{dB}{d\tilde{\psi}} = P(\tilde{\psi}) .} \quad (6.6)$$

As in (6.5), (6.6), it says that the quantity $\frac{v_x - u_y + f}{h}$ is **constant along a streamline**.

With the aid of the relationship $P(\tilde{\psi}) = \frac{dB}{d\tilde{\psi}}$ we can now examine closely the cases

where B is uniform. When both the relative vorticity and f are independently zero (i.e., $v_x - u_y = 0$ and $f = 0$), the potential vorticity P is zero so B is a constant. Also, when the

relative vorticity cancels the planetary vorticity (i.e., $v_x - u_y + f = 0$), P is zero so B is a constant. This latter case is sometimes referred to as the zero potential vorticity case.

Since $v_x = \frac{\partial}{\partial x} \left(\frac{\tilde{\Psi}_x}{h} \right)$ and $u_y = \frac{-\partial}{\partial y} \left(\frac{\tilde{\Psi}_y}{h} \right)$, the p.v. can also be written as:

$$\frac{\partial}{\partial x} \left(\frac{\tilde{\Psi}_x}{h} \right) + \frac{\partial}{\partial y} \left(\frac{\tilde{\Psi}_y}{h} \right) + f = h P(\tilde{\Psi})$$

or

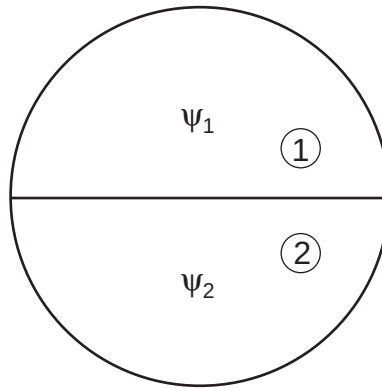
$$\square \bullet \frac{\square \tilde{\Psi}}{\odot h} + f = h P(\tilde{\Psi}) .$$

Note that both the p.v. and Bernoulli **do not** involve any approximations. Their power results from **their particular structure** – the Bernoulli does not involve derivatives and the potential vorticity contains the inertial (nonlinear) terms in a linear form.

This completes the derivation of the Bernoulli and the potential vorticity equation.

VII. Stationary modons

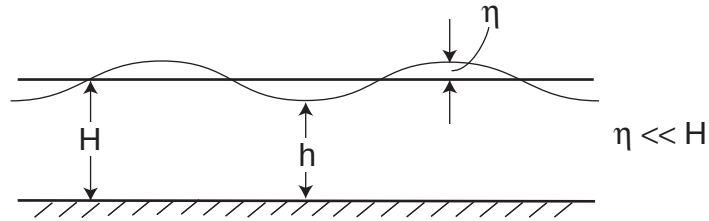
We shall now derive the detailed solution for a stationary modon,



which obeys, of course, the stationarity condition,

$$\beta \iint (\psi_1 + \psi_2) dx dy = 0 . \quad (7.1)$$

Assume that fluid is barotropic with a flat bottom,



and define the streamfunction

$$\psi_x = v \quad ; \quad \psi_y = -u \quad .$$

For this case the potential vorticity

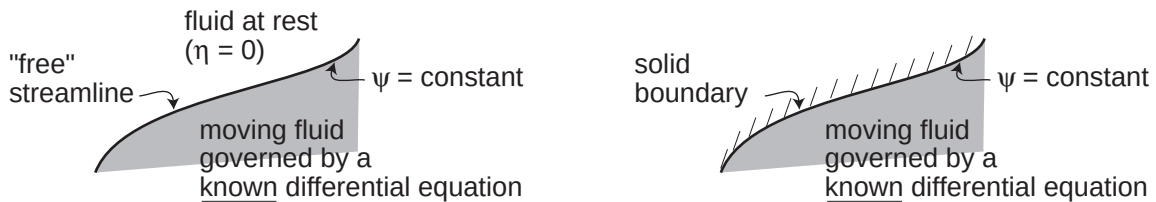
$$v_x - u_y + f = (H + \eta) P(\tilde{\psi})$$

reduces to the vorticity equation

$$\boxed{\nabla^2 \psi + f = P'(\psi) \quad ,} \quad (7.2)$$

where P' is a function of ψ . As mentioned, the power of the equation is that the **nonlinear terms** are now expressed in a **linear** form.

Before proceeding, it is appropriate to recall the b.c. along a free streamline (which should be distinguished from the b.c. along solid boundaries).

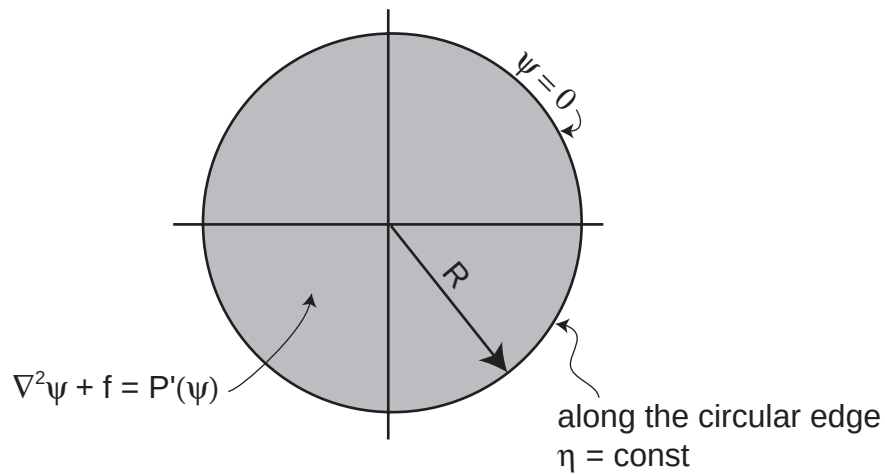


Both situations require ψ to be a constant along the boundary. In the solid boundary case this is a sufficient boundary condition. However, since the position of the free streamline (left panel) is not known in advance (because it is “free” to move), a second b.c. is required in this case. This second condition is that the pressure be constant along it. Via Bernoulli, this gives that the quantity $\frac{1}{2}(u^2 + v^2)$ must be conserved along the line. This second condition compensates for the fact that the position of the free streamline is not

known in advance but rather must be determined as a part of the problem (ref: Batchelor 1969, Garabedian 1964).

We shall restrict the boundary of our particular patch to be a circle which means that, in general, we over-constrain the system. This implies that we need to free something else such as the vorticity which constitutes the differential equation governing the field.

We have, therefore,



Also, along the edge $u^2 + v^2 = \text{constant}$ and P' , the vorticity, is not known *a priori*. As mentioned, the latter compensates for the fact that we took the boundary to be given.

Now use **polar coordinates**

$$\nabla^2 \Psi = \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \Psi$$

and choose the general form of the vorticity to be

$$\boxed{P'(\Psi) = f_0 - \lambda \Psi} \quad ,$$

where λ is to be determined.

We have then,

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \Psi + f_0 + \beta r \sin \theta = f_0 - \lambda \Psi$$

or

$$\boxed{\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}\right) \psi + \lambda \psi = -\beta r \sin \theta} \quad (7.3)$$

subject to the two boundary conditions:

$$\psi = 0 \quad ; \quad r = R \quad ; \quad 0 \leq \theta \leq 2\pi \quad (7.4a)$$

and
$$\frac{\partial}{\partial \theta} \left(\frac{\partial \psi}{\partial r} \right) = 0 \quad ; \quad r = R \quad ; \quad 0 \leq \theta \leq 2\pi \quad . \quad (7.4b)$$

A particular solution satisfying the right-hand side of (7.3) is,

$$\psi_p = -\beta r \sin \theta / \lambda \quad (7.5)$$

{ Proof:

$$\left[\frac{1}{r} \left(\frac{\partial}{\partial r} \right) r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \left(\frac{-\beta r \sin \theta}{\lambda} \right) + \lambda \left(\frac{-\beta r \sin \theta}{\lambda} \right) \\ = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{-\beta r \sin \theta}{\lambda} \right) + \left(\frac{\beta}{\lambda r} \sin \theta \right) - \beta r \sin \theta = -\beta r \sin \theta . \quad \}$$

We now recall that the general solution is,

$$\psi = \psi_H + \psi_p$$

and that

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \psi_H + \lambda \psi_H = 0 . \quad (7.6)$$

Assume that ψ_H is separable, i.e.,

$$\psi_H = F(\theta) G(r) . \quad (7.7)$$

Now substitute (7.7) into (7.6) to get,

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] FG + \lambda FG = 0$$

or

$$\frac{F}{r} \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + \frac{G}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \lambda F G = 0 . \quad (7.8)$$

Now multiply (7.8) by $\frac{r^2}{FG}$ to get,

$$\frac{r}{G} \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + \frac{1}{F} \frac{\partial^2 F}{\partial \theta^2} + \lambda r^2 = 0 \quad (7.9)$$

which, according to the separation of variables principle, can be written as,

$$\frac{r}{G} \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + \lambda r^2 = \frac{-1}{F} \frac{\partial^2 F}{\partial \theta^2} = C , \quad (7.10)$$

where C is a constant. We shall shortly take C to be unity and see later that otherwise a solution cannot exist. Equation (7.10) means that,

$$\frac{\partial^2 F}{\partial \theta^2} + FC = 0$$

so that

$$F = A \sin C^{1/2} \theta + B \cos^{1/2} \theta . \quad (7.10a)$$

Also, (7.10) implies

$$r \frac{\partial}{\partial r} \left(r \frac{\partial G}{\partial r} \right) + G \lambda r^2 - CG = 0 . \quad (7.11)$$

We now take $C = 1$ and define

$$r' = r \lambda^{1/2}$$

so that (7.11) becomes,

$$\frac{r'}{\lambda^{1/2}} \frac{\partial}{\partial r'} \lambda^{1/2} \left(\frac{r'}{\lambda^{1/2}} \frac{\partial G}{\partial r'} \lambda^{1/2} \right) + G(r')^2 - G = 0$$

or

$$r' \frac{\partial}{\partial r'} \left(r' \frac{\partial G}{\partial r'} \right) + G[(r')^2 - 1] = 0 . \quad (7.12)$$

Equation (7.12) can also be written as,

$$(r')^2 \frac{d^2 G}{d(r')^2} + r' \frac{dG}{dr'} + [(r')^2 - 1] G = 0 , \quad (7.13)$$

which is the Bessel equation function of $O(1)$.

The solution of (7.13) is the Bessel function of the first kind and first order.

$$G = J_1(r') = \sum_{K=0}^{\infty} \frac{(-1)^K (r'/2)^{2K+1}}{K! (K+1)!}$$

or

$$G = J_1(r\lambda^{1/2}) = \sum_{K=0}^{\infty} \frac{(-1)^K (r\lambda^{1/2}/2)^{2K+1}}{K! (K+1)!} . \quad (7.14)$$

Combining (7.10a) and (7.14) we have,

$$\psi_H = (A \sin\theta + B \cos\theta) J_1(r\lambda^{1/2})$$

so that the total solution ψ is,

$$\psi = (A \sin\theta + B \cos\theta) J_1(r\lambda^{1/2}) - \beta r \sin\theta/\lambda . \quad (7.15)$$

We shall now deal with the boundary conditions.

The first b.c. is:

$$\psi = 0 ; \quad r = R ; \quad 0 \leq \theta \leq 2\pi$$

which implies that

$$[A \sin\theta + B \cos\theta] J_1(R\lambda^{1/2}) - \beta R \sin\theta/\lambda = 0 .$$

To satisfy this we need,

$$B = 0 ; \quad A \sin\theta J_1(R\lambda^{1/2}) = \beta R \sin\theta/\lambda$$

which gives,

$$A = \frac{\beta R}{\lambda J_1(R\lambda^{1/2})} .$$

Hence, we have,

$$\boxed{\psi = \frac{\beta R \sin\theta}{\lambda J_1(R\lambda^{1/2})} J_1(r\lambda^{1/2}) - \frac{\beta r \sin\theta}{\lambda} .} \quad (7.16)$$

The second boundary condition is,

$$\left. \frac{\partial}{\partial \theta} \left(\frac{\partial \Psi}{\partial r} \right) \right|_{r=R} = 0 \quad .$$

Noting that,

$$\psi_r = \frac{\beta R \sin \theta}{\lambda J_1(R\lambda^{1/2})} \lambda^{1/2} J_1'(r\lambda^{1/2}) - \frac{\beta \sin \theta}{\lambda}$$

we get,

$$\psi_{r\theta}|_{r=R} = \frac{\beta R \cos \theta}{\lambda J_1(R\lambda^{1/2})} \lambda^{1/2} J_1'(R\lambda^{1/2}) - \frac{\beta \cos \theta}{\lambda} = 0$$

which gives,

$$\boxed{R\lambda^{1/2} J_1'(R\lambda^{1/2}) = J_1(R\lambda^{1/2})} \quad (7.17)$$

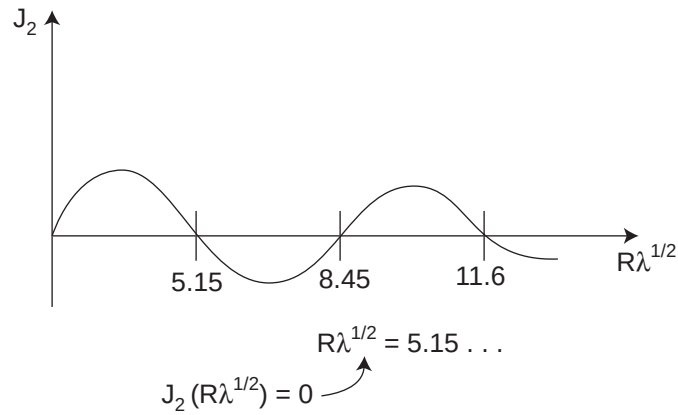
Now use the identity,

$$(R\lambda^{1/2}) J_1'(R\lambda^{1/2}) = J_1(R\lambda^{1/2}) - (R\lambda^{1/2}) J_2(R\lambda^{1/2})$$

which, together with (7.17), gives,

$$\boxed{J_2[R\lambda^{1/2}] = 0} \quad (7.18)$$

This is the desired solution for λ .

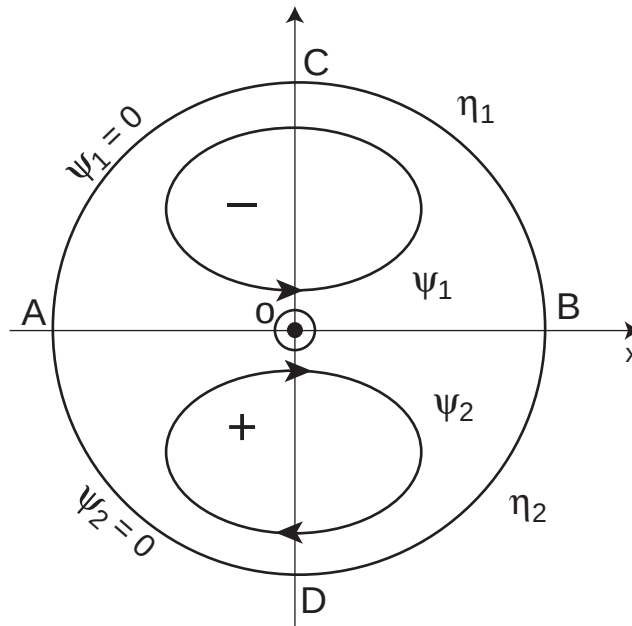


It turns out that the first root gives a two-cell modon whereas the higher roots give a higher number of cells.

Homework #3:

1. Determine and plot the horizontal velocity components (u, v), the streamfunction ψ and the free surface elevation associated with the Stern's "modon" (hint: to compute η , use the Bernoulli).
 2. Determine and plot the structure of the field when the second and third Bessel function zeros of the Stern's modon are considered.
-
-

At some point along our discussion of the "modon," a question arose as to why the integrated equations do not vanish for each cell.



To see this recall,

$$uu_x + vu_y - fv = -g\eta_x$$

$$uv_x + vv_y + fu = -g\eta_y$$

$$u_x + v_y = 0 ,$$

and integrate the y momentum equaiton over the northern cell to get,

$$\beta \iint \psi_1 \, dx dy = g \oint \eta_1 \, dx = g \underbrace{\int_A^B \eta_1 \, dx}_{\text{through 0}} + g \underbrace{\int_B^A \eta_1 \, dx}_{\text{through C}} . \quad (7.19)$$

Since the second term on the right-hand side of (7.19) is zero we get,

$$\beta \iint \psi_1 \, dx dy = g \underbrace{\int_A^B \eta_1 \, dx}_{\text{through 0}} . \quad (7.19a)$$

Similarly, we get for the southern cell,

$$\beta \iint \psi_2 \, dx dy = g \oint \eta_2 \, dx = g \underbrace{\int_A^B \eta_2 \, dx}_{\text{through D}} + g \underbrace{\int_B^A \eta_2 \, dx}_{\text{through 0}} \quad (7.20)$$

which can be written as,

$$\beta \iint \psi_2 \, dx dy = g \int_B^A \eta_2 \, dx = - g \int_A^B \eta_2 \, dx . \quad (7.21)$$

Next, note that since along the separation line the pressure should match, it follows that

$\eta_1 = \eta_2$ along **A0B**. Hence, (7.21) and (7.19a) give,

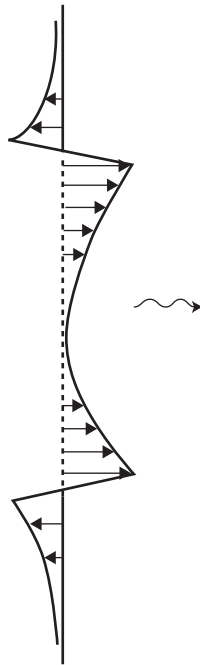
$$\beta \iint (\psi_1 + \psi_2) \, dx dy = 0$$

even though

$$\beta \iint \psi_1 \neq 0 \quad \text{and} \quad \beta \iint \psi_2 \neq 0 .$$

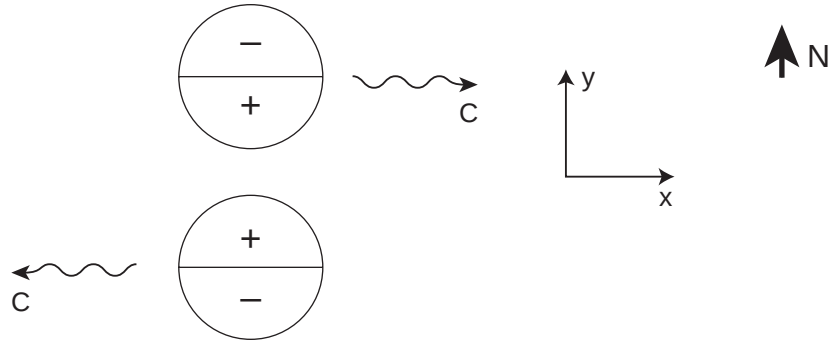
Using Bernoulli, it is easy to show that even though the speed and η are zero at A,B and along the outer boundary **they are not zero along A0B!**

Finally, it is pointed that the modon is stationary on a β plane because the mutual advection of the two cells induces an eastward drift which is exactly cancelled by the westward Rossby wave speed.



VIII. Translating modons

We would now like to look at propagating “modons” because numerical models have shown that modons are **not** always stationary. They indicate that modons move primarily in the zonal direction as shown in the figure below,



To understand this, consider again a barotropic patch with a free surface and write the equations of motions in a **moving** coordinate system. Namely, a coordinate system which travels with the patch at speed **C** (in east-west direction). So, we would like to transform the equations of motion in a **fixed** coordinate system:

$$\left\{ \begin{array}{l} \tilde{u}_{\tilde{t}} + \tilde{u} \tilde{u}_{\tilde{x}} + \tilde{v} \tilde{u}_{\tilde{y}} - (f_0 + \beta \tilde{y}) \tilde{v} = -gh_{\tilde{x}}(1) \\ \tilde{v}_{\tilde{t}} + \tilde{u} \tilde{v}_{\tilde{x}} + \tilde{v} \tilde{v}_{\tilde{y}} + (f_0 + \beta \tilde{y}) \tilde{u} = -gh_{\tilde{y}}(2) \\ \frac{\partial h}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}(h\tilde{u}) + \frac{\partial}{\partial \tilde{y}}(h\tilde{v}) = 0(3) \end{array} \right.$$

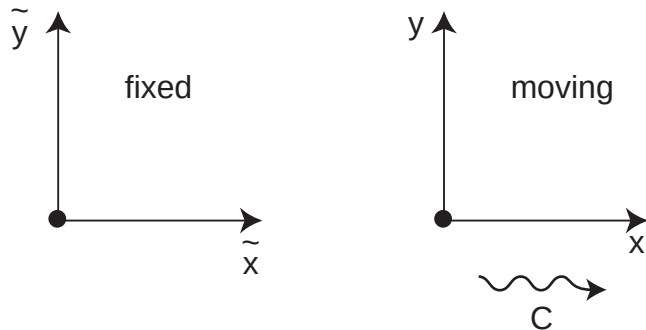
to a moving system. **C**, the migration speed in the **x** direction, is assumed to be constant, namely, all parcels move at the same speed. Also assume that the feature **does not** change its **shape and structure** with time.

The transformation is:

$$x = \tilde{x} - C\tilde{t}$$

$$y = \tilde{y}$$

$$t = \tilde{t}$$



where each of the old variables is now a function of all the new variables, i.e.,

$$\tilde{t} = \tilde{t} (x,y,t)$$

$$\tilde{x} = \tilde{x} (x,y,t)$$

$$\tilde{y} = \tilde{y} (x,y,t) .$$

Using the chain rule one gets,

$$\begin{aligned} \frac{\partial}{\partial \tilde{t}} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial \tilde{t}} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \tilde{t}} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \tilde{t}} \\ &\quad \uparrow \\ &\quad 0 \\ \frac{\partial}{\partial \tilde{x}} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial \tilde{x}} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \tilde{x}} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \tilde{x}} \\ &\quad \downarrow \quad \downarrow \\ &\quad 0 \quad 0 \\ \frac{\partial}{\partial \tilde{y}} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial \tilde{y}} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \tilde{y}} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \tilde{y}} , \\ &\quad \uparrow \\ &\quad 0 \end{aligned}$$

so that,

$$u = \tilde{u} - c$$

$$v = \tilde{v}$$

and (1–3) become,

$$-C \frac{\partial}{\partial x} (u + C) + (u + C) \frac{\partial}{\partial x} (u + C) + v \frac{\partial}{\partial y} (u + C) - (f_0 + \beta y)v = -gh_x$$

$$-C \frac{\partial v}{\partial x} + (u + C) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + (f_0 + \beta y) (u + C) = -gh_y$$

$$-C \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} [h(u + C)] + \frac{\partial}{\partial y} (hv) = 0 ,$$

where,

$$\frac{\partial C}{\partial x} = \frac{\partial C}{\partial y} = 0 \quad \text{since } C \text{ is uniform for all parcels.}$$

Hence,

$$- \mathcal{C}u_x + uu_x + \mathcal{C}u_x + vu_y - (f_0 + \beta y)v = -g h_x$$

$$- \mathcal{C}v_x + uv_x + \mathcal{C}v_x + vv_y - (f_0 + \beta y)(u + C) = -g h_y$$

$$- Ch_x + (u + C)h_x + hu_x + \frac{\partial}{\partial y}(hv) = 0$$

and we get the desired equations,

$uu_x + vu_y - (f_0 + \beta y)v = -g h_x$ $uv_x + vv_y + (f_0 + \beta y)(u + C) = -g h_y$ $\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$

where u, v are now measured relative to the **new** moving coordinates system.

As before, we now integrate again the momentum equation in the y direction multiplied by h ,

$$\frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2) + (f_0 + \beta y)uh + (f_0 + \beta y)Ch = \frac{-g}{2} \frac{\partial}{\partial y}(h^2) \quad (1)$$

and define $uh = -\psi_y$; $vh = \psi_x$.

Recall that Stokes' theorem gives,

$$\iint (M_x - N_y) = \oint Ndx + Mdy$$

so that (1) can be written as,

$$- \oint hv^2 dx + \oint huv dy - \iint f_0 \psi_y dx dy - \beta \iint y \psi_y dx dy + C \iint (f_0 + \beta y)h = (g/2) \oint h^2 dx$$

or,

$$\oint hv(-v dx + u dy) + f_0 \oint \psi dx$$

$$-\beta \iint \left[\frac{\partial}{\partial y} (y\psi) - \psi \right] + C \iint (f_0 + \beta y) h \, dx dy = (g/2) \oint h^2 dx$$

$$\downarrow$$

$$0$$

when we define $\psi = 0$ along the boundary of the patch.

Since the boundary is a streamline $-vdx + udy = 0$ and we get,

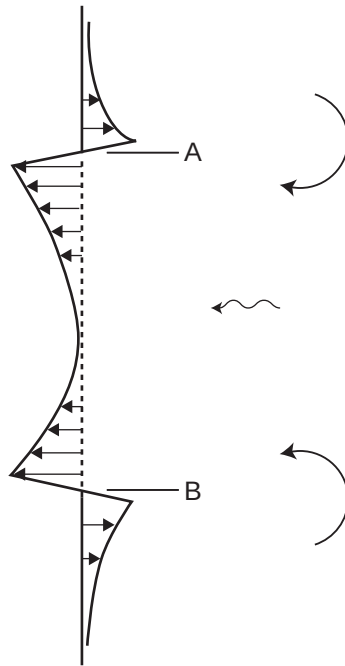
$$\beta \iint \psi \, dx dy + C \iint (f_0 + \beta y) h \, dx dy = (g/2) \oint (H + \eta)^2 \, dx ,$$

where $h = H = \eta$. For $\eta \ll H$ (i.e., rigid lid) we get,

$\beta \iint \psi \, dx dy + C \iint (f_0 + \beta y) h = gH \oint \eta \, dx$		
$\uparrow$ beta force	$\uparrow$ Coriolis force I	$\uparrow$ outside pressure force II

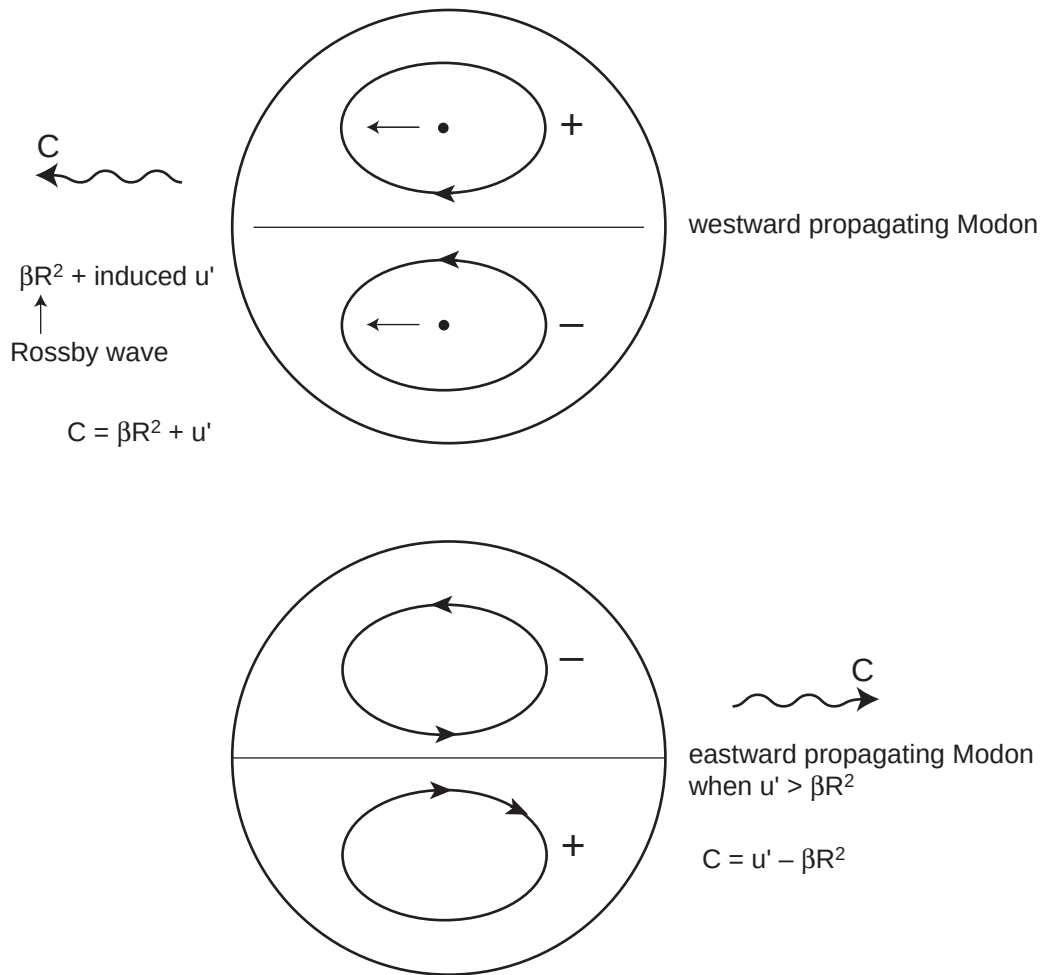
We see that with $I = II = 0$ we recover the stationarity condition and Stern's (stationary) modon. However, when the beta force is zero there can still be a Coriolis force balancing the outside pressure force, i.e., the modon can be moving.

We have shown here with the integrated balance of forces that we can also have **propagating** modons (see Mied and Lindemann 1982, JPO) and not only stationary modons. To understand this more clearly, recall that **linear** vortices can be superimposed on each other,



Linear superposition of two vortices of opposite signs. The system drifts to the left due to the mutually induced speeds.

In the case of the modons, this super-position is imposed on a general Rossby wave tendency to push any feature westward. We can, therefore, have either a westward propagating modon or an eastward propagating modon as shown below,



A stationary modon ($C = 0$) will be present only when the eastward movement tendency is **canceled** by the westward tendency.

Homework #4:

Plot the trajectories of a circular “cloud” of solid particles on a β plane (say, 100 or 1000).

An analytical solution for propagating barotropic modons was derived by Larichev and Reznick (1978). It can be briefly described as follows:

The quasi-geostrophic vorticity equation with a rigid lid is,

$$\frac{\partial}{\partial t}(\square^2 \psi) + \beta \psi_x + J(\psi, \square^2 \psi) = 0$$

where

$$J(a,b) = a_x b_y - a_y b_x$$

and

$$\begin{array}{ccc} x' & = & x - Ct \\ \uparrow & & \uparrow \\ \text{new} & & \text{old} \end{array}$$

is the transformation from the fixed (no prime) to the moving (prime) system. Note that the notation here is according to the “Russian way,” i.e., the equations in the **moving** system are written in terms of the old variables! The equation can also be written as,

$$\nabla^2 \psi = -k^2 (\psi + Cy) - \beta y \quad r < a$$

$$\nabla^2 \psi = -\rho^2 (\psi + Cy) - \beta y \quad r > a ,$$

where a is the radius of the modon and

$$\rho^2 = \beta/C .$$

The boundary conditions are the continuity of the streamfunction,

$$\psi + Cy \Big|_{r=a\pm 0} = 0, \quad \psi + Cy \Big|_{r=a\pm 0}$$

and velocity,

$$\frac{\partial \psi}{\partial r} \Big|_{r=a-0} = \frac{\partial \psi}{\partial r} \Big|_{r=a+0}.$$

The solution is,

$$\psi = \left[A J_1(kr) - \frac{r}{k^2} (\beta + Ck^2) \right] \sin \phi ; r < a$$

$$\psi = B K_1(\rho r) \sin \phi ; r > a ,$$

where

$$A = \frac{\beta a}{k^2 J_1(ka)} ; B = \frac{-Ca}{K_1(\rho a)} .$$

$\uparrow$ $\uparrow$
 Bessel function MacDonald function

This has $C > 0$ solution only, i.e., the feature can only be moving eastward. With a free surface, however, the solution is,

$$\psi = \left[A J_1(kr) - \frac{r^2}{k^2} \left(\beta + \frac{f_0^2 C}{g'H} + Ck^2 \right) \right] \sin\phi ; r < a$$

$$\psi = B K_1(\rho r) \sin\phi ; r > a ,$$

where

$$B = -Ca/K_1(\rho a)$$

$$A = \left(\beta + \frac{f_0^2 C}{g'H} \right) / k^2 J_1(ka) ,$$

and

$$\frac{J_1'(ka)}{J_1(ka)} = \frac{1}{ka} \left(1 + \frac{k^2}{\rho^2} \right) - \frac{k}{\rho} \frac{K_1'(\rho a)}{K_1(\rho a)} .$$

It has the solution:

$$C > 0 \quad \text{or} \quad C < -\frac{\beta g'H}{f^2} ,$$

implying that the feature can either be moving eastward or westward.

Homework #5:

Plot ψ and depth contours for the Larichev and Reznick propagating modon:

$$\psi = \left[A J_1(kr) - \frac{r^2}{k^2} \left(\beta + \frac{f_0^2 C}{g'H} + Ck^2 \right) \right] \sin\phi ; r < a \quad \textcircled{1}$$

$$\psi = B K_1(\rho r) \sin\phi ; r > 0 \quad \textcircled{2}$$

$$B=-Ca/K_1(\rho a)$$

$$A=\left(\beta+\frac{f^2C^2}{gH}\right)/\;k^2J_1(ka)$$

$$\rho^2=[\beta+f_0^2C/gH]/C\;.$$

The original note by Larichev and Reznik is reproduced below.

TWO-DIMENSIONAL ROSSBY SOLITON:
AN EXACT SOLUTION

by V. D. Larichev and G. M. Reznik

The problem of the physical mechanism that can maintain two-dimensional solitary eddies is being extensively discussed these days. The exact solution we suggest here shows that the joint influence of β -effect and nonlinearity can lead to two-dimensional isolated Rossby waves.

The equation for a stream function corresponding to a plane motion of uniform fluid on a β -plane is

$$\frac{\partial \Delta \psi}{\partial t} + \beta \frac{\partial \psi}{\partial x} + J(\psi, \Delta \psi) = 0; \quad (1)$$

$$J(a, b) = a_x b_y - a_y b_x.$$

Assuming that $\psi = \psi(x - Ut, y)$, $U > 0$, and using coordinates $x' = x - Ut, y$, equation (1) is transformed into

$$J(\psi + Uy, \Delta \psi + \beta y) = 0. \quad (2)$$

We divide the plane x', y into two parts: a circle of radius with a center at the point $x' = y = 0$, and an outer region. Let ψ satisfy the equations

$$\Delta \psi = -\kappa^2 (\psi + Uy) - \beta y \quad \text{if } r < a; \quad (3)$$

$$\text{and} \quad \Delta \psi = -\rho^2 (\psi + Uy) - \beta y \quad \text{if } r > a; \quad (4)$$

where $\kappa > 0$, $\rho > 0$, and $r = \sqrt{x'^2 + y^2}$. In such a case, one can readily see that the function ψ satisfies equation (2) in the whole plane (except perhaps the line $r = a$).

We require ψ to decrease as rapidly as the exponential for $r \rightarrow \infty$. It then follows from equation (4) that

$$\rho^2 = \beta/U. \quad (5)$$

(The parameters κ, a are arbitrary for the present.) Using polar coordinates r, ϕ ($x' = r \cos \phi$, $y = r \sin \phi$), we consider the following class of particular solutions of equations (3) and (4).

$$\psi = (AJ_1(\kappa r) - \frac{r}{\kappa^2} (\beta + U\kappa^2)) \sin\phi \quad \text{if } r < a ;$$

$$\psi = BK_1(\rho r) \sin\phi \quad \text{if } r > a ;$$

where J_1 , K_1 are the Bessel function of the first kind and the MacDonald function, respectively. One can easily determine constants A,B, requiring $\Delta\psi$ to be continuous across a line $r = a$. This means an absence of a vorticity source along that circumference and can be transformed by use of equations (3) and (4) into

$$\psi + Uy \Big|_{r = a \pm 0} = 0 . \quad (7)$$

Thus, one can find

$$A = \frac{\beta a}{\kappa^2 J_1(\kappa a)} ; B = - \frac{Ua}{K_1(\rho a)} .$$

If we require in addition

$$\frac{\partial \psi}{\partial r} \Big|_{r = a-0} = \frac{\partial \psi}{\partial r} \Big|_{r = a+0} \quad (8)$$

we see that all functions $\psi, \frac{\partial \psi}{\partial \phi}, \frac{\partial \psi}{\partial r}, \frac{\partial^2 \psi}{\partial \phi^2}, \frac{\partial^2 \psi}{\partial \phi \partial r}, \frac{\partial^2 \psi}{\partial r^2}$, are continuous together with pressure gradient across a line $r = a$. It follows from equation (8) that

$$\frac{J_1'(\kappa a)}{J_1(\kappa a)} = \frac{1}{\kappa a} \left(1 + \frac{\kappa^2}{\rho^2} \right) - \frac{\kappa}{\rho} \frac{K_1(\rho a)}{K_1'(\rho a)} . \quad (9)$$

It is known that

$$\frac{J_1'(\kappa a)}{J_1(\kappa a)} = \frac{1}{\kappa a} + 2\kappa a \Re_n \left(\kappa^2 a^2 - \gamma_{1,n}^2 \right)^{-1} ,$$

where $\gamma_{1,n}$ is a growing sequence of zeros of the function $r^{-1}J_1(r)$.

From this and equation (9), one can readily see that for any fixed a, ρ , equation (9) has a discrete number of roots κ . Thus, the conditions of equations (7) and (8) are satisfied and therefore the velocity and pressure fields are continuously differentiated, but the vorticity field is continuous only since $\frac{\partial^3 \psi}{\partial r^3}$ has a jump at $r = a$. It is important to notice that because of the finite value of this jump, our solution satisfies equation (2) over the whole plane. Figure 4 schematically shows streamlines (in coordinates $x' = x - Ut, y$), corresponding to minimum κa for fixed a, ρ .

A remarkable feature of the solution is the rapid decay proportional to $\frac{1}{\sqrt{r}} \exp [-\sqrt{\beta/U^1} - r]$ as $r \rightarrow \infty$. This solution represents an isolated eddy formation moving in quiescent fluid, and it is natural therefore to identify it as a two-dimensional solitary Rossby wave.

One can show that if $\beta = 0$, such a wave (with an exponential decay) is impossible. On the other hand, in the linear case of equations (1) and (2), rapidly decaying solutions without singularities do not exist. Thus, the existence of a two-dimensional solitary Rossby wave is a result of the joint influence of the β -effect and nonlinearity.

Since $\beta > 0$ and we need positive ρ^2 , equation (5) means that the phase velocity $U > 0$. For $U < 0$ we have $\rho^2 < 0$, and the solution does not decay as the exponential (but exists as in the linear case when $-\kappa^2 = \rho^2 = \beta/U$, giving a decay $\sim r^{-1/2}$). In other words, constructed solitary Rossby waves move to the east in contrast with periodic waves, but this limitation appears in the "rigid lid" approximation only. In the case of a free surface case, the vorticity equation has the following form:

$$\frac{\partial}{\partial t} \left(\Delta \psi - \frac{f_0^2}{gh} \psi \right) + \beta \frac{\partial \psi}{\partial x} + J(\psi, \Delta \psi) = 0, \quad (10)$$

where g is the acceleration due to gravity, and h is the depth. Supposing again that $\psi = \psi(x - Ut)$, one can reduce equation (10) to the same form as equation (2):

$$J(\psi + Uy, \Delta \psi + \beta' y) = 0; \quad \beta' = \beta + \frac{f_0^2 U}{gh}. \quad (11)$$

It is evident that the equation constructed above is simultaneously a solution of equation (11) and decays as an exponential for $r \rightarrow \infty$ if $\rho^2 = \beta'/U > 0$. This last condition imposes the following limitation on a phase velocity:

$$U > 0 \quad \text{or} \quad U < -\frac{\beta_0 gh}{f_0^2}. \quad (12)$$

Thus, solitary Rossby waves can also move to the west ($U < 0$), but with significant (in the barotropic case) velocity.

A final conclusion about the physical reality of these solutions can only be made after consideration of their stability.

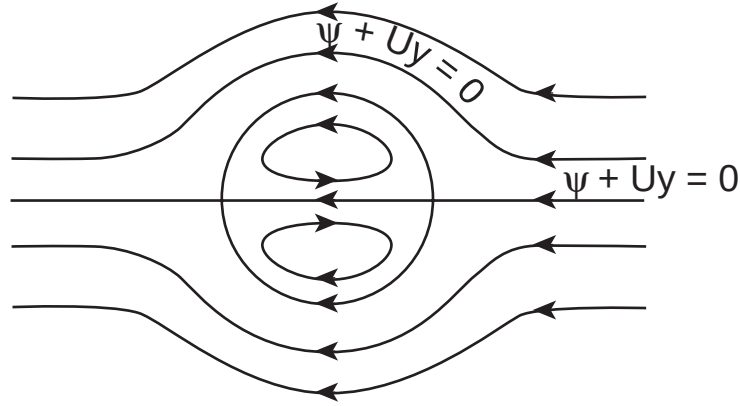
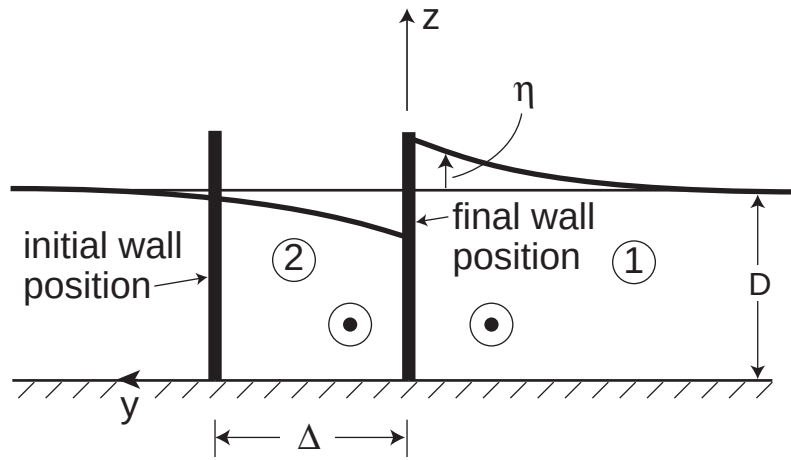


Figure 4. Schematic streamlines; isolines of the function $\psi(x', y) + Uy$. (Larichev and Reznik)

IX. Introduction to the Rossby's adjustment problem

Consider first a wall pushed horizontally a distance Δ in a barotropic fluid:



a. *Linear momentum:*

The x momentum equation is,

$$\frac{du}{dt} - fv = -g \frac{\partial h}{\partial x}, \quad (1)$$

where $h = D + \eta$. We note, however, that,

$$v + \frac{dy}{dt} = 0 \quad ; \quad \text{and} \quad \frac{\partial h}{\partial x} = 0$$

so that (1) gives,

$$\frac{du}{dt} - f \frac{dy}{dt} + 0 \quad (2)$$

or:

$$\boxed{\frac{d}{dt}(u - fy) + 0} \quad (3)$$

Application of (3) to the fluid attached to the wall gives,

$$\boxed{-f\Delta = \hat{U}}$$

for the speed next to the wall.

b. *Potential vorticity and geostrophy:*

$$\frac{-u_y + f}{D + \eta} = \frac{f}{D} \quad \text{and} \quad fu = -g\eta_y \quad \Leftrightarrow \quad u_{yy} - \frac{u}{R_d^2} = 0$$

and

$$\eta_{yy} - \frac{\eta}{R_d^2} = 0.$$

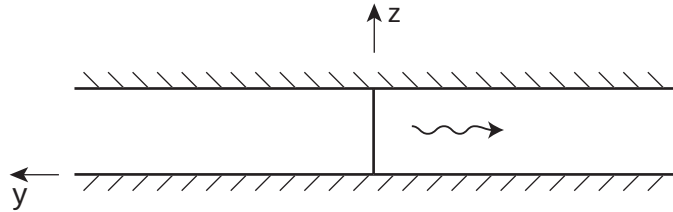
The solution satisfying $u_1 \rightarrow 0$ at $t \rightarrow \infty$ and $u_2 \rightarrow 0$ at $y \rightarrow \infty$ is

$$u_1 = -f\Delta e^{y/R_d} \quad ; \quad y < 0$$

$$u_2 = -f\Delta e^{-y/R_d} \quad ; \quad y > 0$$

where $R_d^2 = gD/f^2$.

For completeness, consider now the fully barotropic case.



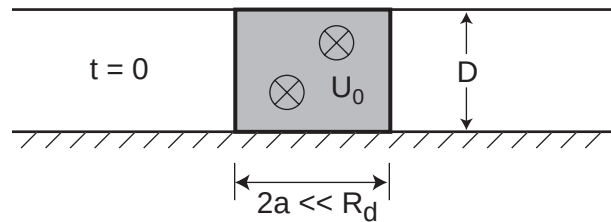
Here, $R_d \rightarrow \infty$ so that $u = -f\Delta$ is the speed everywhere.

With the aid of this simple example we shall now go to the complete adjustment problem.

X. The classical Rossby adjustment problem

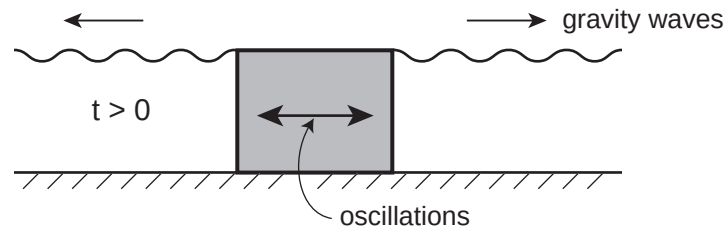
(For a good reference see: Morel, *Dynamical Meteorology*, 1973.)

Initially ($t = 0$) there is an unbalanced uniform and unidirectional current occupying a width $2a$ as shown below:

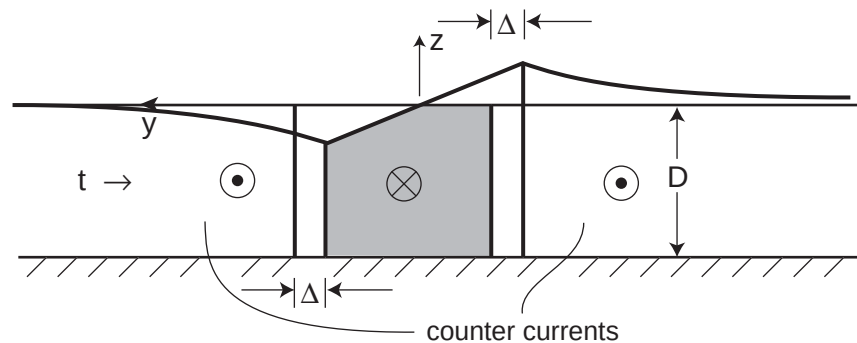


For simplicity we take $a \ll R_d$, i.e., the current is much narrower than the Rossby radius.

When rotation is “turned on” the entire current is deflected to the right and gravity waves are generated due to zonal oscillations of the current’s position.



After a period of $O(f^{-1})$ a steady state where the current weakens and is displaced to the right of its original position is established:

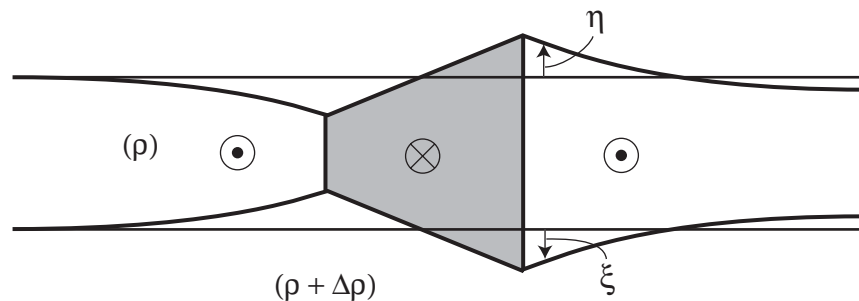


Importance of result: Any unbalanced system will try to reach a geostrophic balance. Potential vorticity is conserved even if energy is not, provided, of course, that there is no friction so that energy is lost by other mechanisms (such as gravity waves).

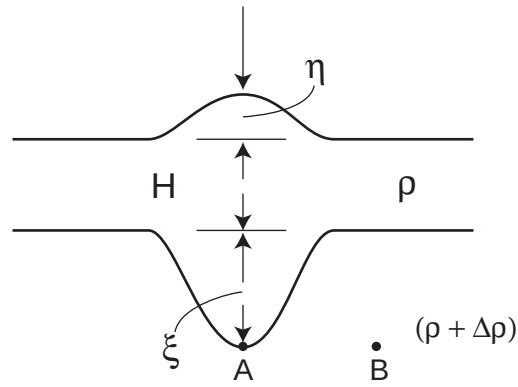
Homework #6:

Derive the detailed analytical solution for Rossby's adjustment problem and calculate the energy loss.

Rossby then extended his barotropic result to a two-layer system with the lower layer motionless. The final adjusted state looks like,



For this case, the relationship between the interface level ξ and the free sea-level displacement η is found from the condition of no-flow (i.e., no pressure gradients) in the lower layer,



Namely, we use the condition

$$P_A = P_B$$

which gives

$$(\eta + H + \xi) \rho = \xi(\rho + \Delta\rho)$$

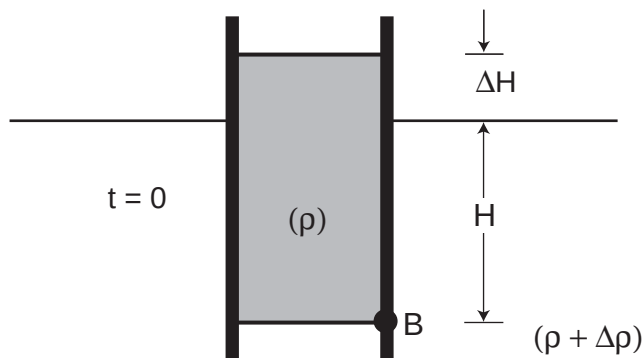
so that

$$\boxed{\eta = \frac{\Delta\rho}{\rho} \xi}.$$

Since usually $(\Delta\rho/\rho) \ll 1$, $\eta \ll \xi$ and $h = \eta + \xi + H \approx H + \xi$.

XI. Wedge collapse

Consider now the collapse of an **infinitely long wedge** (created by the removal of the walls).



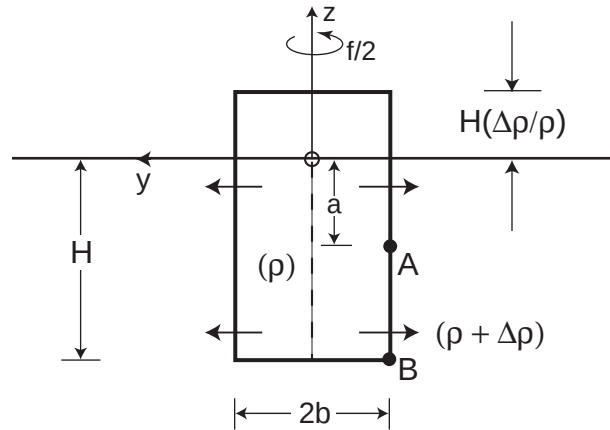
Since the interface does not move at $t = 0$ equalization of the outside and inside pressures at B gives,

$$(\Delta H + H)\rho = H(\rho + \Delta\rho)$$

or,

$$\boxed{\Delta H = H (\Delta\rho/\rho)} ,$$

i.e., at point B the pressure inside the wedge is the same as the pressure outside the wedge. However, at all points above B (say, at A shown below) the pressure inside is greater than the pressure outside, i.e.,



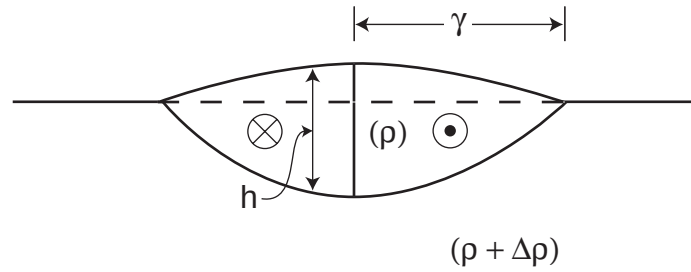
$$P_{A(\text{inside})} - P_{A(\text{outside})} = \left(H \frac{\Delta\rho}{\rho} + a\right) g\rho - g(\rho + \Delta\rho)a$$

$$= Hg\Delta\rho + a \rho - \rho a - g\Delta\rho a$$

$$= g\Delta\rho (H - a) > 0$$

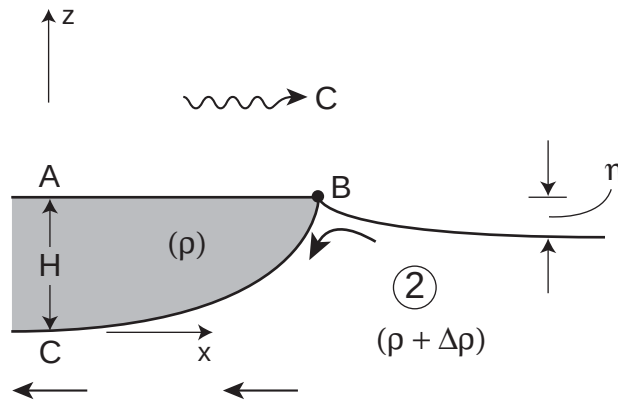
since $a < H$.

When $f \rightarrow 0$ no force can balance the pressure difference so the fluid is continuously spreading horizontally. With rotation, however, a steady state with a **geostrophic** balance will be reached after a period of $O(f^{-1})$.



a. *How fast is the spreading?*

To see how fast the front (i.e., the line corresponding to $h = 0$) will move horizontally, consider, for simplicity, a steady propagation without rotation ($f \equiv 0$).



Recall that the Bernoulli in 3D is:

$$\frac{1}{2} (u^2 + v^2 + w^2) + \frac{P}{\rho} + gz = \text{const along a streamline} .$$

Application to fluid ② along BC gives,

$$\underbrace{gH}_{\text{point B}} = \underbrace{\frac{C^2}{2\rho} + \frac{P}{\Delta\rho}}_{\text{point C}} = \frac{C^2}{2\rho} + \frac{gH\rho}{\Delta\rho}$$

$$= \frac{C^2}{2} + gH - gH \frac{\Delta\rho}{\rho}$$

$$\frac{C^2}{2} = g'H$$

$$C = \sqrt{2 g' H} .$$

We see that the speed is the nonlinear internal wave speed. Note, however, that the above calculation is only an estimate because we have not really shown that the problem has a steady solution. (It turns out that it does not.)

(Now back to the infinitely long wedge)

b. Governing equations and general solution

We would like to find the final steady state without solving the complete time-dependent problem. As before, since there is no friction, the conservation of p.v. gives:

$$\frac{f}{H(1 + \Delta\rho/\rho)} = \frac{-u_y + f}{h} ,$$

but since $\Delta\rho/\rho \ll 1$, it can be approximated by,

$$\frac{f}{H} \approx \frac{-u_y + f}{h} . \quad (3)$$

Now geostrophic balance implies,

$$fu = -g'h_y . \quad (4)$$

The assumption of **no flow** in the lower layer gives $\eta = \frac{\Delta\rho}{\rho} h$

so that $g\eta_y \approx g \frac{\Delta\rho}{\rho} h_y = g'h_y$.

We now take the derivative of (3) with respect to y,

$$\frac{f}{H} h_y = -u_{yy} .$$

and combine with (4) to get,

$$u_{yy} - \frac{u}{R_d^2} = 0 ,$$

where $Rd = (g'H^{1/2}/f)$ is the Rossby deformation radius.

c. Detailed solution:

The general solution is: $u = Ae^{y/Rd} + Be^{-y/Rd}$

and from (3),

$$h = H \left(1 - \frac{A}{fR_d} e^{y/Rd} + \frac{B}{fR_d} e^{-y/Rd} \right). \quad (5)$$

The first “boundary condition” is associated with the symmetry of the problem.

Specifically, we expect the solution to be symmetrical so that $h(-y) = h(y)$, or,

$$\begin{aligned} h &= H \left(1 - \frac{A}{fR_d} e^{y/Rd} + \frac{B}{fR_d} e^{-y/Rd} \right) \\ &= H \left(1 - \frac{A}{fR_d} e^{-y/Rd} + \frac{B}{fR_d} e^{y/Rd} \right) \\ \frac{e^{y/Rd}}{fR_d} (-A - B) &= \frac{e^{-y/Rd}}{fR_d} (-A - B). \end{aligned}$$

Hence,

$$-A - B = 0 \quad ; \quad A = -B.$$

The second boundary condition is that the **cross-section** area is conserved, i.e.,

$$bH = \int_0^\gamma h dy,$$

where γ is the current's width which is to be determined as part of the problem.

Computation of the cross-sectional area gives:

$$\begin{aligned} bH &= \int_0^\gamma H \left(1 - \frac{A}{fR_d} e^{y/Rd} + \frac{B}{fR_d} e^{-y/Rd} \right) dy \\ &= H \left(y - \frac{A}{f} e^{y/Rd} - \frac{B}{f} e^{-y/Rd} \right) \Bigg|_0^\gamma \end{aligned}$$

$$= H\left(\gamma - \frac{A}{f} e^{\gamma/R_d} - \frac{B}{f} e^{-\gamma/R_d}\right) - H\left(-\frac{A}{f} - \frac{B}{f}\right) = 0$$

$$bH = H\left[\gamma - \frac{A}{f} (e^{\gamma/R_d} - e^{-\gamma/R_d})\right]$$

$$b = \left[\gamma - \frac{A}{f} \left(\delta - \frac{1}{\delta}\right)\right], \quad (6)$$

where

$$\delta = e^{\gamma/R_d}.$$

Note that this is identical to the condition coming from the conservation of linear momentum discussed earlier,

$$\frac{d}{dt}(u - fy) = 0.$$

The third boundary condition is,

$$h = 0 \quad ; \quad y = \gamma.$$

It gives,

$$1 - \frac{A}{fR_d} e^{\gamma/R_d} - \frac{A}{fR_d} e^{-\gamma/R_d} = 0$$

$$1 - \frac{A}{fR_d} \left(\delta + \frac{1}{\delta}\right) = 0.$$

Therefore,

$$A = fR_d / \left(\delta + \frac{1}{\delta}\right)$$

which can be combined with (6) to give,

$$b = \left[\gamma - R_d \left(\frac{\delta^2 - 1}{\delta^2 + 1}\right)\right]$$

so that

$$b = \left[\gamma - R_d \left(\frac{e^{2\gamma/R_d} - 1}{e^{2\gamma/R_d} + 1}\right)\right].$$

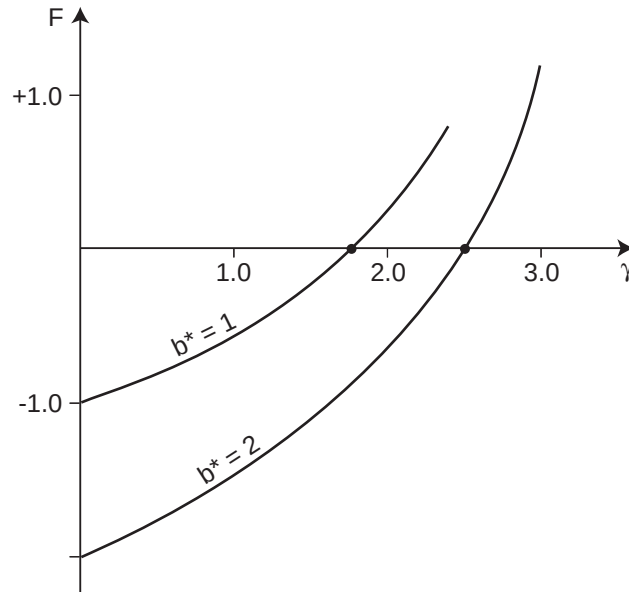
For convenience, we nondimensionalize the above relationship to get

$$b^* = \frac{b}{R_d} = \left[\frac{\gamma}{R_d} - \frac{e^{2\gamma/R_d} - 1}{e^{2\gamma/R_d} + 1}\right] = \gamma^* - \frac{e^{2\gamma^*} - 1}{e^{2\gamma^*} + 1}$$

or

$$F = \gamma^* - \frac{e^{2\gamma^*} - 1}{e^{2\gamma^*} + 1} - b^* = 0 .$$

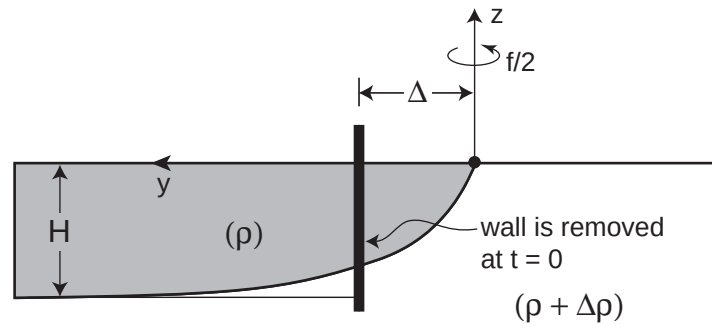
It is easy to obtain the solution (to this algebraic relationship) graphically as it corresponds to the intersections of the curves (shown below) with the γ axis.



We saw in this problem that there is a redundancy of some of the conditions that we used, that adjustment is reached in a period of $O(f^{-1})$ and that, in the presence of rotation, disturbances move only a distance of the Rossby radius away from their origination point.

XII. A broad collapse

To better understand the problem, consider now another example for the redundancy of some of the conditions. At $t = 0$, a wall is removed as shown.



a. *PV and geostrophy*

The equations are:

$$-u_y + f = h \frac{f}{H}$$

$$fu = -g'hy \text{ .}$$

As before, they give $u_{yy} - u/R_d^2 = 0$ where $R_d^2 = g'H/f^2$.

b. *Solution:*

$$u = Ae^{-y/R_d}$$

Application of the boundary condition $h = 0$; $y = 0$ to

$$h = H \left(1 + \frac{A}{fR_d} Ae^{-y/R_d} \right)$$

gives

$$A = -fR_d \text{ .}$$

Also, applicaiton of linear momentum conservation

$$\frac{d}{dt} (u - fy) + 0$$

gives

$$u = -\sqrt{g'H} e^{-y/R_d} \text{ ; } h = H (1 - e^{-y/R_d})$$

so that

$$\sqrt{g'H} = 0 - f\Delta \text{ and } \Delta = (g'H)^{1/2}/f \text{ .}$$

This means that the displacement Δ is exactly equal to the Rossby radius.

Check now conservation of the volume:

$$\int_0^{\Delta} h dy = \int_{\Delta}^{\infty} (H - h) dy$$

$$\int_0^{\Delta} H (1 - e^{-y/R_d}) dy = \int_{\Delta}^{\infty} H e^{-y/R_d} dy$$

$$\int_0^{\Delta} H dy - \int_0^{\Delta} H e^{-y/R_d} dy = \int_{\Delta}^{\infty} H e^{-y/R_d} dy$$

$$H \Delta = - R_d H e^{-y/R_d} \Big|_0^{\infty}$$

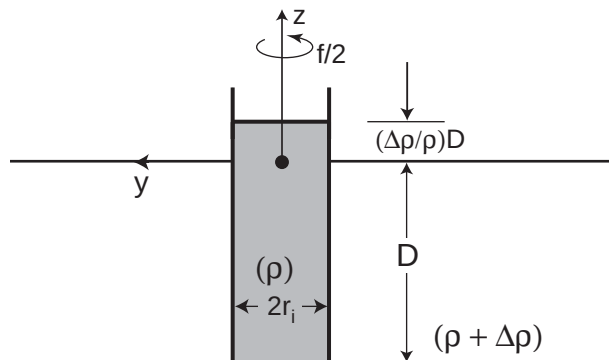
which gives

$$\Delta = R_d ,$$

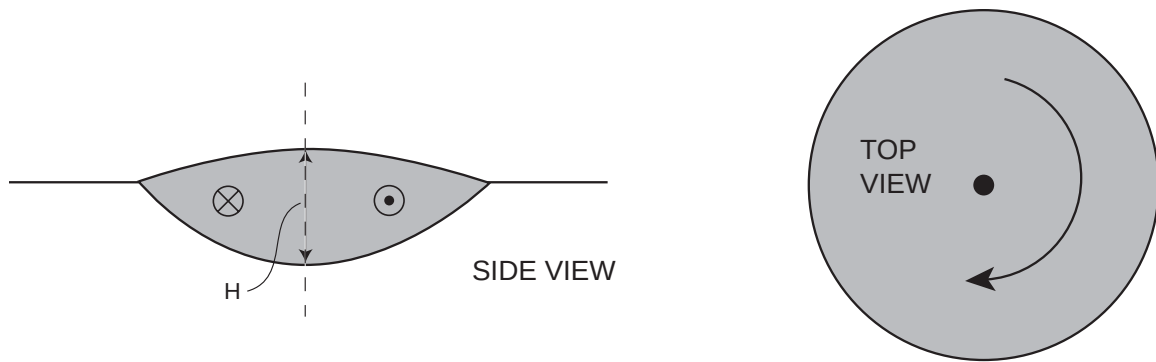
showing a redundancy of the volume conservation constraint with the linear momentum conservation.

XIII. Collapse of a cylinder

Now, we look at an equivalent process, the collapse of a cylinder.



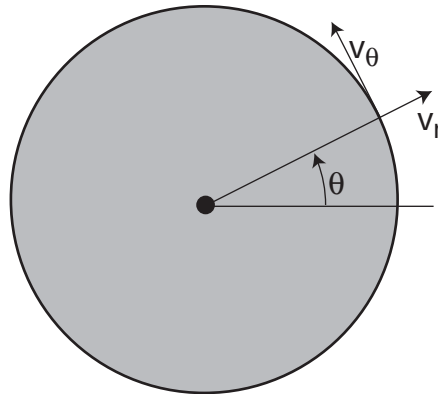
Expect to find an anticyclonic eddy,



First, derive the conservation of angular momentum for an axisymmetric flow. To do so, use the momentum equation in the θ direction,

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + f v_r = 0, (1)$$

where the velocities are defined by the figure below,



and $\partial/\partial\theta = 0$.

Recall that $\frac{D}{Dt} + \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta}$,

and multiply (1) by r to get,

$$\frac{\partial}{\partial t} (r v_\theta) + r v_r \frac{\partial v_\theta}{\partial r} + v_\theta v_r + f r v_r = 0$$

or
$$\frac{\partial}{\partial t} \left(r v_\theta + \frac{f r^2}{2} \right) + v_r \frac{\partial}{\partial r} \left(r v_\theta + \frac{f r^2}{2} \right) = 0$$

which gives:

$$\boxed{\frac{D}{Dt} \left(r v_\theta + \frac{f r^2}{2} \right) + 0} . \quad (2)$$

This is analogous to the one-dimensional linear momentum considered earlier,

$$\boxed{\frac{D}{Dt} (u - f y) = 0} .$$

Getting back now to the cylindrical collapse problem, for $r_i \ll R_d$ (when $R_d^2 = g'D/f^2$) we immediately find from (2) that

$$v_\theta = - f r / 2$$

because at $t = 0$, $v_\theta = 0$ and $r = 0$.

Now consider the p.v. equation for the final adjusted state, i.e., take $\frac{\partial}{\partial \theta} \neq 0$ and $v_r \rightarrow 0$, to get,

$$\frac{1}{r} \frac{d}{dr} (r v_\theta) + f = h \frac{f}{D} . \quad (3)$$

The momentum equation is,

$$\frac{v_\theta^2}{r} + f v_\theta = g' h_r . \quad (4)$$

Assume next that $h/D \rightarrow 0$ (i.e., $D \rightarrow \infty$). This assumption is justified when $r_i \ll R_d$. Note that this condition is frequently referred to as the **zero p.v. case**. To see when is this limit valid, note that conservation of volume implies that

$$\pi r_0^2 D \sim \frac{g'H}{f^2} H .$$

where H is the maximum depth of the final lens. For $D \rightarrow \infty$, the p.v. equation becomes,

$$\frac{1}{r} \frac{d}{dr} (r v_\theta) + f = 0$$

$$\frac{d}{dr}(rv_\theta) = -fr$$

$$rv_\theta = -\frac{fr^2}{2}$$

$$v_\theta = -\frac{fr}{2}, \quad (4a)$$

which is identical to the solution obtained earlier using the conservation of angular momentum.

Now substitute (4a) into the momentum equation to get,

$$\frac{f^2 r^2}{4r} - \frac{f^2 r}{2} = g'h_r$$

or, $g'h_r = -\frac{f^2 r}{4}.$

Integrate in r to get, $g'h = H - \frac{f^2 r^2}{8}$

or, $\boxed{h = H - \frac{f^2 r^2}{8g'}},$

where, as before, H is the central depth of the eddy. The radius of the eddy corresponds to $h = 0$, i.e.,

$$R = \sqrt{8g'H} / f = 2\sqrt{2} R_{df},$$

where $R_{df} = (g'H)^{1/2}/f$ is now defined on the basis of maximum depth at the **center**.

Now, if $\mathbf{D} \sim \mathbf{h}$, namely the cylinder is **not** infinitely deep, then,

$$\frac{1}{r} \frac{d}{dr}(r_\theta) + f = \frac{hf}{D} \quad (5)$$

and $\underbrace{\frac{v_\theta^2}{r}} + \underbrace{fv_\theta}_{=} = g' \frac{dh}{dr} \quad (6)$

since $I/II \sim O(R_0)$ which at times can be assumed to be small, we may try to use the approximation:

$$fv_0 \approx g' \frac{dh}{dr}$$

which gives

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{g'}{f_0} \frac{dh}{dr} \right) + f_0 - \frac{hf_0}{D} = 0$$

or,

$$\frac{d^2h}{dr^2} + \frac{1}{r} \frac{dh}{dr} - \frac{h}{R_d^2} + \frac{D}{R_d^2} = 0$$

where

$$R_d = (g'D)^{1/2}/f_0 ,$$

i.e., the Rossby radius is defined again on the basis of the **initial** depth.

Now define $r' = r/R_d$,

$$\frac{1}{R_d^2} \frac{d^2h}{d(r')^2} + \frac{1}{R_d^2 r'} \frac{dh}{dr'} - \frac{h}{R_d^2} + \frac{D}{R_d^2} = 0 .$$

and multiply by $(r')^2$ to get,

$$(r')^2 \frac{d^2h}{dr'^2} + r' \frac{dh}{dr'} - (h - D) (r')^2 = 0 .$$

Note that

$$h = h_p + h_H ,$$

where h_p is the particular solution and h_H is the homogeneous solution.

We immediately see that the particular solution is,

$$h_p = D ,$$

so that the equation for the homogeneous solution is,

$$(r')^2 \frac{d^2 h_H}{d(r')^2} + r' \frac{dh_H}{dr'} - h_H (r')^2 = 0 .$$

This is like the Bessel equation: $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (p^2 + x^2)y = 0$

with $p = 0$.

The solution is,

$$h_H = A I_0 (r/R_d) + B K_0 (r/R_d)$$

$\uparrow$
 r'

where A, B are constants to be determined from the boundary conditions.

Here, I — modified Bessel function of the first kind

K — modified Bessel function of the second kind

We won't need the complete definition of K_0 because $K_0 \approx -\log(r/R_d)$ for $r/R_d \rightarrow 0$,

implying that $B = 0$.

The total solution is now,

$$h = h_H + h_p = A I_0 (r/R_d) + D .$$

Since $h = 0$ at, say, $r = R$,

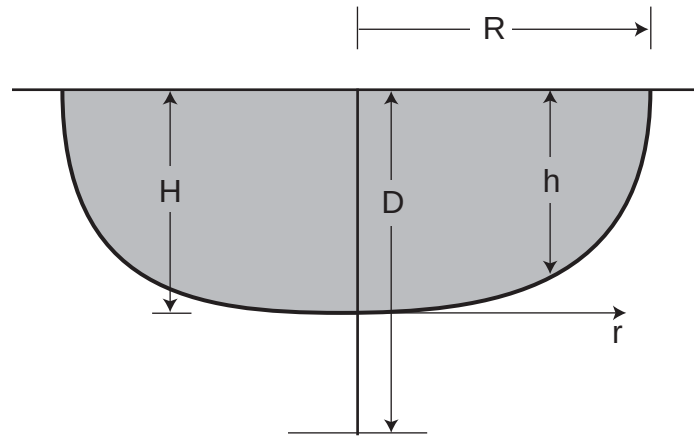
$$h = 0 = D + A I_0 (R_0/R_d)$$

$$A = -\frac{D}{I_0(R_0/R_d)}$$

so that the solution is:

$$h = D \left[1 - \frac{I_0(r/R_d)}{I_0(R/R_d)} \right] ,$$

showing that $h < D$ at $r = 0$ as illustrated below,



The tangential velocity v_θ is,

$$v_\theta = \frac{-g'}{f} \frac{dh}{dr}$$

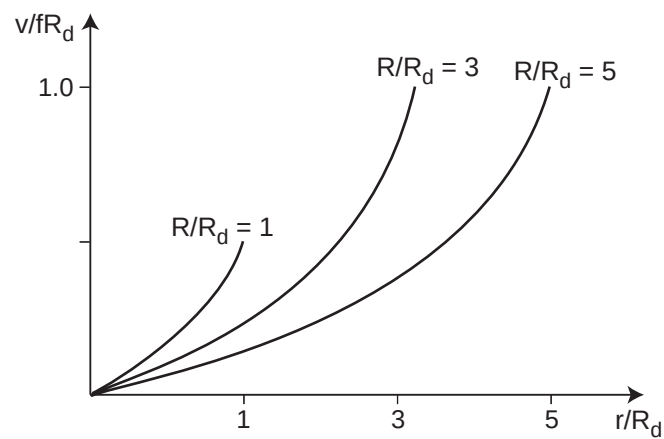
$$v_\theta = -\frac{g'H}{f I_0(R/R_d)} \frac{d}{dr} [I_0 (r/R_d)]$$

$$= -\frac{g'H I_1(r/R_d)}{R_d f I_0(R/R_d)} = \boxed{-f R_d \frac{I_1(r/R_d)}{I_0(R/R_d)}}$$

where we have used the identity

$$\frac{d}{dx} [I_0 (\alpha x)] = \alpha I_1(\alpha x) .$$

The velocity profile looks like,



Typically,

$$\Delta\rho/\rho \sim 2 \times 10^{-3} \ ; \ H \sim 500 \text{ m}$$

$$f \sim 10^{-4} \text{sec}^{-1} \quad ; \quad R_d \sim 30 \text{ km}$$

so that $v \sim O(\alpha f R_d) \sim (\alpha 10^{-4} \times 30,000) \sim 3 \text{ m/sec}$

$$\boxed{v \sim 1 \text{ m/sec}} .$$

But the solution breaks down at $h = 0$ (along the edge) because there,

$$\frac{1}{r} \frac{d}{dr} (r v_\theta) + f = 0$$

so that

$$v_\theta = -fr/2 \quad \Big|_{r=R}$$

and

$$\frac{v_\theta^2}{r} \sim \frac{1}{2} f v_\theta ,$$

implying that the geostrophic approximation **is not** accurate along the rim.